Lecture 32 Proof of Riemann mapping theorem

RMT : Suppose $\Omega \neq \mathbb{C}$ is a simply connected domain. There is a mapping $f: \Omega \to D_1(0)$ with f analytic, one-to-one, and onto. [Ω is "conformally equivalent to the unit disc."]

Lemma 1 There exists $h: \Omega \to D_1(0)$ that is analytic and one-to-one.

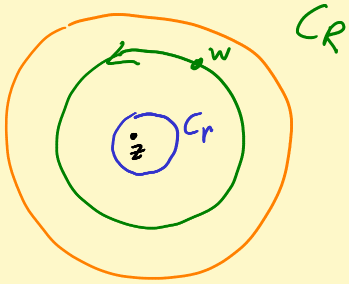
Pf Later. Easy consequence of Fact: If G is analytic and non-vanishing on a simply connected domain Ω, there is an analytic g on Ω with $g^2 = G$ on Ω.

Lemma 2 If f_n analytic on domain Ω and $f_n \rightrightarrows f$ on Ω, then $f_n^{(k)} \rightrightarrows f^{(k)}$ too!

Way false $\mathbb{R} \to \mathbb{R}$: $h_N(t) = \sum_{n=1}^{N} \frac{1}{2^n} \sin(2^n t)$ ← $h_N \rightrightarrows$ cont nowhere diff'ble fcn on $\mathbb{R}$ (Weierstraß)

or $f_n(t) = \frac{1}{n} \sin(2^n t)$ easier

Pf



$$|f_n^{(k)}(z) - f^{(k)}(z)|$$

$$= \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \, dw \right|$$

$$\leq \frac{k!}{2\pi} \left(\max_{C_R} |f_n - f| \right) \frac{1}{(R-r)^{k+1}} \cdot 2\pi R \quad \checkmark$$

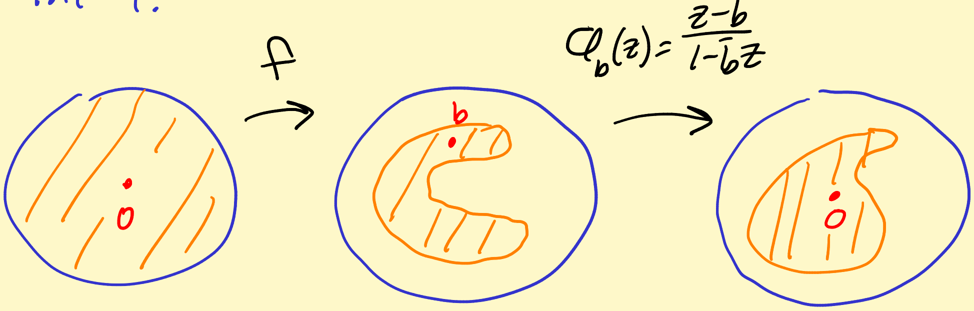
$\to 0$ by unif conv on compact C_R.

Given $K \subset\subset \Omega$, cover K by blue discs like above. Take finite subcover.

Lemma 3 (Schwarz #3) $f : D_1(0) \circlearrowleft$ analytic.

Then $|f'(0)| \le 1$. If $|f'(0)| = 1$, then $f(z) = \lambda z$, λ const with $|\lambda| = 1$.

Pf



Schwarz : $|(\varphi_b \circ f)'(0)| \le 1$

$$\underbrace{|\varphi_b'(b)|}_{\frac{1}{1-|b|^2}} \, |f'(0)| \le 1$$

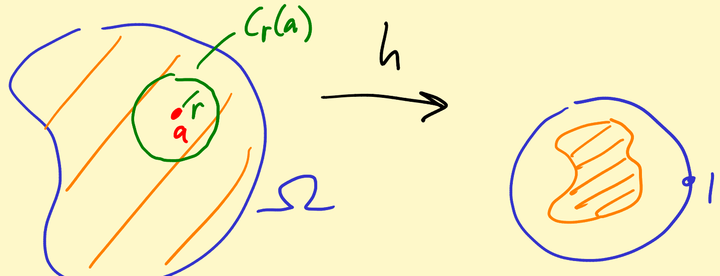
So $|f'(0)| \le 1 - |b|^2 \le 1$. ✓

If $|f'(0)| = 1$, then $b = 0$. $f(0) = 0$. Have Schwarz. ✓

Pf of RMT Let $\tilde{\mathcal{F}}$ = family of analytic fcns on Ω mapping one-to-one into $D_1(0)$. Lemma 1 $\Rightarrow \tilde{\mathcal{F}} \neq \emptyset$.

Pick $a \in \Omega$. Let $M = \sup\{|h'(a)| : h \in \tilde{\mathcal{F}}\}$.

Step 1 $M < \infty$.



$\exists \ \overline{D_r(a)} \subset \Omega$. Cauchy estimate : $|h'(a)| \le \dfrac{1! \max_{C_r(a)} |h|}{r^1} \le \dfrac{1}{r}$

Remark. Can let $r \nearrow \text{dist}(a, b\Omega)$. $|h'(a)| \le \frac{1}{\text{dist}(a, b\Omega)}$

Step 2 Take $h_n \in \tilde{\mathcal{F}}$ with $|h_n'(a)| \to M$.

Since $|h_n| < 1$ on Ω, Montel's $\Rightarrow$ $\exists$ subseq $h_{n_k} \rightrightarrows F$ on Ω. Note: Know F is analytic.

Claim F is Riemann map.

Step 3 Lemma 2 $\Rightarrow$ $h_{n_k}'(a) \to F'(a)$.

So $|F'(a)| = M$.

Note: $|F'(a)| = M \neq 0$ because 1-1 analytic fcns have non-vanishing derivatives. So F is not constant.

So Hurwicz #2 $\Rightarrow$ F is one-to-one on Ω.

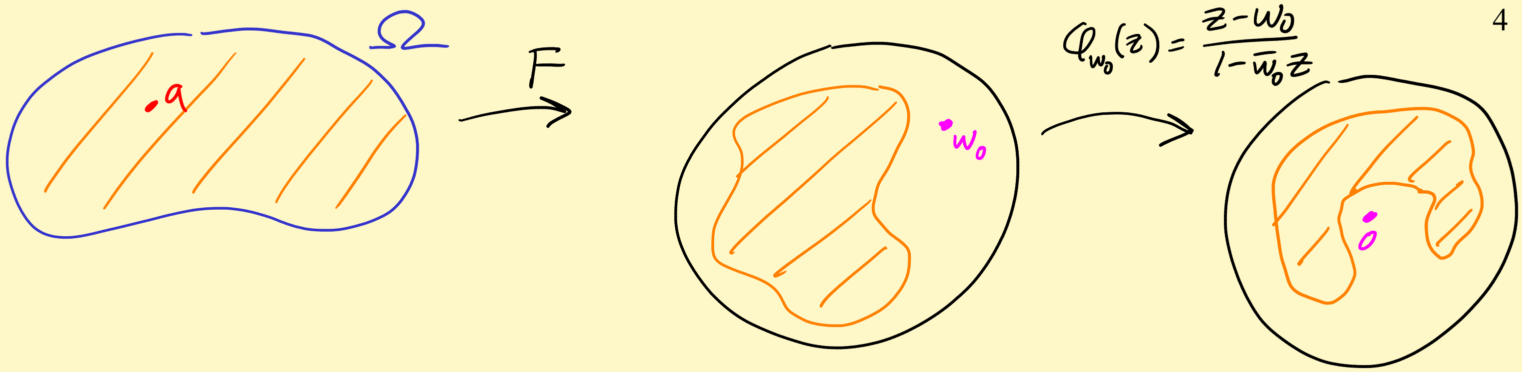
Step 4 Claim $F \in \tilde{\mathcal{F}}$. Just need to show F maps into $D_1(0)$. Easy: $|h_{n_k}(z)| < 1$

So $|F(z)| = |\lim_{k\to\infty} h_{n_k}(z)| \le 1$.

But, if $|F(z_0)| = 1$, max princ $\Rightarrow$ $F \equiv$ const of modulus 1. ↯. So $F: \Omega \to D_1(0)$. ✓

So $M = \text{Max}\{|h'(a)| : h \in \tilde{\mathcal{F}}\}$.

Step 5 F is onto $D_1(0)$. Suppose not. $\exists\, w_0 \in D_1(0) - F(\Omega)$



Notice that $\varphi_{w_0} \circ F$ is <u>non-vanishing</u> on Ω. So

$\exists$ analytic square root fcn $g(z)$ on Ω with

$$g(z)^2 = \varphi_{w_0}(F(z))$$

<u>Note</u> $|g(z)|^2 < 1 \Rightarrow |g(z)| < 1$ on Ω.

So $g : \Omega \to D_1(0)$.

<u>Claim</u> g is 1-1, so $\in \mathcal{F}$.

<u>Why</u>

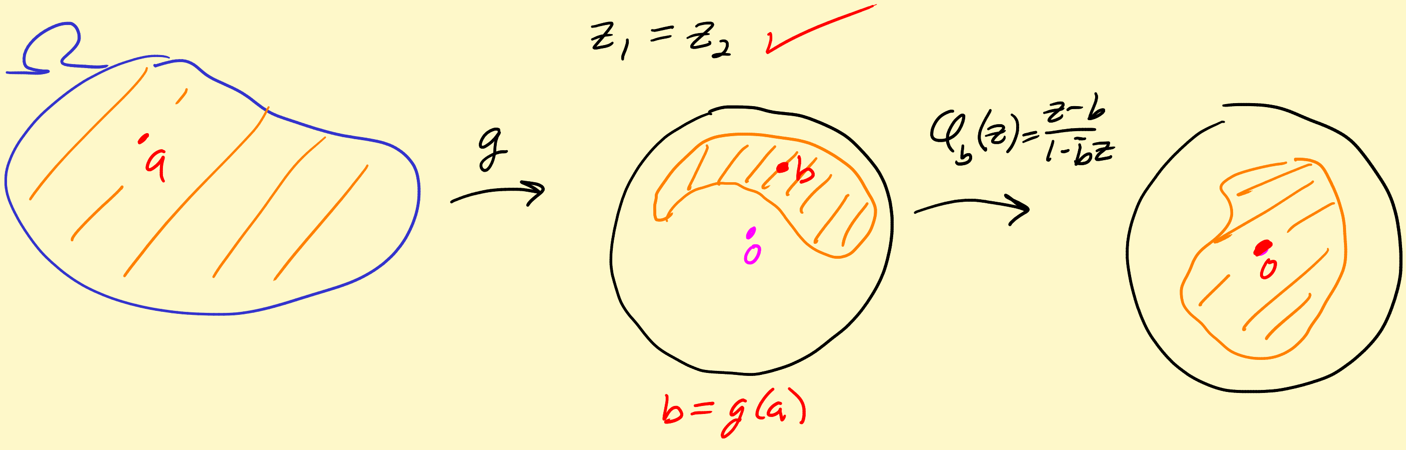
$$g(z_1) = g(z_2)$$

$$g(z_1)^2 = g(z_2)^2$$

$$\varphi_{w_0}(F(z_1)) = \varphi_{w_0}(F(z_2)) \leftarrow \varphi_{w_0} \text{ 1-1}$$

$$F(z_1) = F(z_2) \leftarrow F \text{ 1-1}$$

$$z_1 = z_2 \ \checkmark$$



Drum roll! $\varphi_b \circ g \in \tilde{\mathcal{F}}$ and

$$|(\varphi_b \circ g)'(a)| > M \quad ⚡$$

So no such w_0 exists! F is onto. ✓

Why Schwarz: $\tilde{F} = \varphi_b \circ g$

Check that $F = G \circ \tilde{F}$ where

$$G = \underbrace{\varphi_{w_0}^{-1} \circ s \circ \varphi_b}_{\text{analytic, } D_1(0) \circlearrowleft} \quad \text{where } s(z) = z^2.$$

Lemma 3 (Schwarz #3) $\Rightarrow |G'(0)| \le 1$

and $|G'(0)|$ can't $= 1$, because G is not 1-1 because $s(z)$ is not, so $G(z) \not\equiv \lambda z$.

So $\boxed{|G'(0)| < 1}$

Finally $F'(a) = G'(\underbrace{\tilde{F}(a)}_{=0})\,\tilde{F}'(a)$

$$\underbrace{|F'(a)|}_{=M} = \underbrace{|G'(0)|}_{\substack{<1 \\ \neq 0}}\ \underbrace{|\tilde{F}'(a)|}_{\text{must be } >M} \quad ⚡.$$

Done!