Thm: f analytic on open set $\Omega \implies$ C-R Eqns (and first partials exist).

Ex: $f(z) = \bar{z} = x - iy$

$u(x,y) = x$ $\quad v(x,y) = -y$

Don't satisfy C-R Eqns.
So $\bar{z}$ not analytic

Basic facts about diff'tion $\mathbb{R} \to \mathbb{R}$ hold $\mathbb{C} \to \mathbb{C}$ for complex derivatives: $(fg)' = f'g + fg'$, etc.

Exercise: Prove chain rule in complex case. Adapt Baby Rudin proof to $\mathbb{C} \to \mathbb{C}$.

$E(x+iy) = e^x \cos y + i e^x \sin y$

Properties:
0) $E(z)$ is entire
1) $E'(z) = E(z)$
2) $E(0) = 1$
3) $E(z+w) = E(z)E(w)$
4) $E(z) \neq 0$ for all z
5) $E(-z) = 1/E(z)$
6) $E(z-w) = E(z)/E(w)$
7) $E(x) = e^x$ for $x \in \mathbb{R}$
8) $E(i\theta) = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$
9) $E(z) = 1 \iff z = n2\pi i$, $n \in \mathbb{Z}$.

Remark: Easy to cook up $E(z)$ from 0-2:

Suppose $E(z)$ analytic, $E(0)=1$, $E'(z)=E(z)$

$$E'(z) = \begin{cases} \#1 \ \boxed{u_x + i v_x} \\ \quad\quad \Vert \\ \#2 \ \boxed{v_y - i u_y} \end{cases} = u + iv$$

$\#1$: $u_x = u \leftarrow u = \lambda(y) e^x$

$v_x = v \leftarrow v = \sigma(y) e^x$

$\#2$: $v_y = \sigma'(y) \cdot e^x = \lambda(y) \cdot e^x = u_x$

$u_y = \lambda'(y) e^x = -\sigma(y) \cdot e^x = -v_x$

$$\boxed{\begin{matrix} \sigma' = \lambda & A \\ \lambda' = -\sigma & B \end{matrix}}$$

A': $\sigma'' = \lambda' \underset{\underset{B}{\uparrow}}{=} -\sigma \leftarrow$ see $\sigma'' + \sigma = 0$

also get $\lambda'' + \lambda = 0$

$E(0)=1$ gives sin and cos for σ and λ.

Lemma: Suppose f is analytic on $D_r(a)$ and $f' \equiv 0$ there. Then f must be constant.

Pf: $f = u + iv$ $\quad\quad f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \equiv 0$

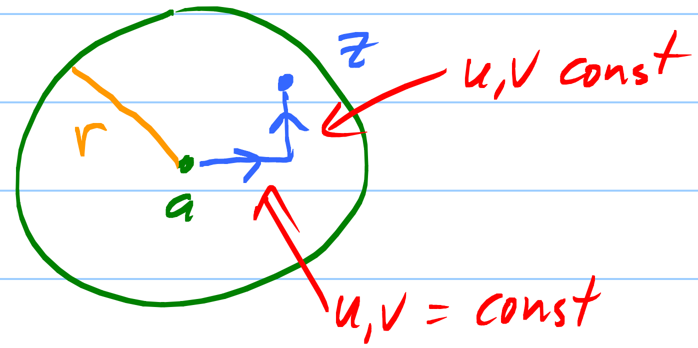
See all first partials of u and v $= 0$, i.e., $\nabla u \equiv 0$ and $\nabla v \equiv 0$.

Freshman Calculus: Suppose $h(t)$ is continuous on $[a,b]$ and h' exists on (a,b) and $h'(t) \equiv 0$

on (a,b). Then $h \equiv$ const on $[a,b]$. Pf: MVT

Pf of complex case: f analytic $\Rightarrow$ f contin. $\Rightarrow$ u,v contin. Also, f analytic $\Rightarrow$ first partials exist.

Use one var version on blue lines!



Lemma: If $f(z)$ is analytic on $D_R(0)$ and $f'(z) = k f(z)$ there. Then $f(z) = C\,E(kz)$ where $C = f(0)$.

Pf: Suppose $f' - kf = 0$ $(*)$

ODE instincts: Multiply by $E(-kz)$:

Note that $\frac{d}{dz}\left(E(-kz)\right) = E'(-kz)\frac{d}{dz}(-kz)$

chain rule $= E(-kz)\cdot(-k)$.

$(*)\cdot E(-kz)$: $f' E(-kz) - k E(-kz) f(z) \equiv 0$

$$\frac{d}{dz}\left[f(z)E(-kz)\right] \equiv 0$$

Lemma $\Rightarrow$ $\boxed{f(z)E(-kz) \equiv C}$, a const.

Stop! Start proof over using $f(z) = E(kz)$.

Note that $f'(z) = E'(kz)\frac{d}{dz}(kz) = kE(kz)$
$= k\,f(z)$ ✓

Get $\boxed{E(kz)E(-kz) \equiv C}$ Let $z=0$ to see $C = \underline{1}$.

Aha! $E(kz)$ cannot be zero. (#4 proved)

Also, $E(-kz) = 1/E(kz)$ (#5 proved)

Back to proof above: $f(z)E(-kz) \equiv C$.

Let $z=0$ to see $C = f(0)$. Finally

$$f(z) = C\left(1/E(-kz)\right) = CE(kz)$$ ✓

Proof of #3: $E(z+w) = E(z)E(w)$

Sneaky trick: Let $f(z) = E(z+w)$. ← w = const.

Then $f'(z) = E'(z+w)\underbrace{\frac{d}{dz}(z+w)}_{1} = E(z+w) = f(z)$

Lemma $\Rightarrow$ $f(z) = \underbrace{f(0)}_{E(0+w)}E(\underset{k=1}{1}\cdot z) = E(w)E(z)$

$$E(z+w) = E(z)E(w)$$ ✓

Rest follow!

From now on, write $E(z) = e^z$.

Big Fact: f analytic $\Rightarrow f'$ analytic $\Rightarrow f''$ analytic $\Rightarrow f''' \dots$ etc.

Pf: Soon.

Consequence 1: $f = u + iv$ analytic $\Rightarrow u, v$ C^∞-smooth!

Consequence 2: f analytic $\Rightarrow u, v$ harmonic.

Pf of 2 (assuming Big Fact):

C-R Eqns $\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$ $\quad \frac{\partial}{\partial x}(A)$: $u_{xx} = v_{xy}$

$\frac{\partial}{\partial y}(B)$: $u_{yy} = -v_{yx}$

$(+)$ $\Delta u = 0$

using calculus fact: v C^2-smooth $\Rightarrow v_{xy} = v_{yx}$.

Also, $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ shows v harmonic. ✓

Def^n: u is harmonic means 1) u is C^2-smooth, and 2) $\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\Delta u} = 0$.

Important Fact: Given a harmonic function u

on a disc $D_r(a)$, there is another harmonic function v there such that $u+iv$ is analytic on $D_r(a)$. v is called a harmonic conjugate for u.

Question: Is v unique?

Hint: C-R Eqns: v_1 and v_2, $\nabla v_1 \equiv \nabla v_2$.

v_1, v_2 differ by a constant.