Complex series

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ of complex numbers a_n converges, then $a_n \to 0$ as $n \to \infty$.

Pf: Suppose $\sum_{n=0}^{\infty} a_n = L \in \mathbb{C}$. Let $S_N = \sum_{n=0}^{N} a_n$. Note that $\lim_{N\to\infty} S_N = L$ and that $a_N = S_N - S_{N-1} \longrightarrow L - L = 0$ as $n \to \infty$.

Def^n: A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n| < \infty$.

Lemma: Absolutey convergent series converge.

Power Series: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

Big Fact: Power series have a radius of convergence $R \geq 0$, meaning

1) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely for $z \in D_R(z_0)$.
2) converges uniformly on $D_r(z_0)$ for $0 < r < R$.
3) diverges if $|z-z_0| > R$.

Hadamard's Formula: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

Def^n: $\limsup_{n\to\infty} r_n = \lim_{N\to\infty} \underbrace{\sup\{r_n : n \geq N\}}_{\text{non-increasing seq, so limit exists}}$

(could be $-\infty$).

Note: $r_n = |a_n|^{1/n} \geq 0$. So $\frac{1}{R} \geq 0$.

($R=\infty$ case corresponds to $\limsup = 0$.)

Later: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

What is $\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}}$?

Stirling's Formula:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)$$

$$\text{So} \quad \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Note: n-th root of 1st ineq shows

$$\underbrace{\left[\frac{n}{e}(2\pi n)^{\frac{1}{2n}}\right]}_{\to\infty} < \sqrt[n]{n!}, \quad \text{So } \frac{1}{\sqrt[n]{n!}} \to 0$$

as $n\to\infty$.

So $\sum \frac{z^n}{n!}$ has R. of C. $= \infty$.

<u>Geometric Series:</u> $\sum_{n=0}^{\infty} z^n$.

$$S_N = 1 + z + z^2 + \cdots + z^N$$
$$z S_N = \quad z \quad z^3 + \cdots + z^N + z^{N+1}$$
$$(1-z) S_N = 1 + 0 + \cdots + 0 - z^{N+1}$$

$$S_N = \frac{1 - z^{N+1}}{1-z} = \underbrace{\frac{1}{1-z}}_{\text{limit}} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

$|z+w| \le |z| + |w|$ ← numerator estimate

$|z-w| \ge \big||z| - |w|\big|$ ← denominator estimate

or $|z+w| \ge \big||z|-|w|\big|$ too.

Suppose $|z| < r < 1$. $|E_N(z)| = \dfrac{|z|^{N+1}}{|1-z|} \le \dfrac{|z|^{N+1}}{\big|1-|z|\big|} =$

$$\frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-r} \xrightarrow[N\to\infty]{} 0 \text{ if } r<1.$$

So $\sum z^n$ converges unif. on $D_r(0)$, $r<1$.

Note: If $|z|>1$, then $|z^n|>1$. Aha! the z^n don't go to zero. So series does not converge. So R. of C. $= 1$. ✓ Hadamard.

Note: $\sup_{z \in D_1(0)} |E_N(z)| = \infty$. $\left(E_N(t) = \frac{t^{N+1}}{1-t} \quad \lim_{t \to 1^-} E_N(t) = +\infty \right)$

Do not get unif conv on $D_1(0)$.

Pf of Hadamard: Case $R \neq 0$, $R \neq \infty$. Can assume $z_0 = 0$ [via a change of var $w = z - z_0$].
Suppose $0 < r < R$. Pick ρ with $r < \rho < R$.
Consider z with $|z| < r < \rho < R$.

$$\frac{1}{R} < \frac{1}{\rho}$$

$\sup_{n \geq N} \sqrt[n]{|a_n|}$ sliding down to $\frac{1}{R}$

So there is an N_0 such that

$$\sup_{n \geq N_0} \sqrt[n]{|a_n|} < \frac{1}{\rho}.$$

So $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ if $n \geq N_0$.

n-th power: $|a_n| < \frac{1}{\rho^n}$

$$|a_n z^n| = |a_n| |z|^n < |a_n| r^n < \left(\frac{r}{\rho}\right)^n$$

<1!

By comparing geom series $\sum \left(\frac{r}{\rho}\right)^n$, see $\sum a_n z^n$ is <u>absolutely conv</u>.

Next: Get unif conv on $D_r(0)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Same N_0 as above.

Assume $N > N_0$.

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| = \left| \sum_{n=N+1}^{\infty} a_n z^n \right| \leq$$

$$\sum_{n=N+1}^{\infty} \underbrace{|a_n||z|^n}_{< \left(\frac{r}{\rho}\right)^n} = \frac{\left(\frac{r}{\rho}\right)^{N+1}}{1 - \frac{r}{\rho}} \longrightarrow 0$$

as $N \to \infty$.

Last Part: Get divergence when $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup_{n \geq N} \sqrt[n]{|a_n|} \quad \text{for all } N.$$

So, for each N, there is a n such that

that $$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$1 < |a_n z^n|$$

Aha! $a_n z^n$ do not tend to zero. Series diverges.

Remark: $R = 0$, $R = \infty$ similar. Exercise.

EX: $\sum_{n=0}^{\infty} n!\, z^n$ only converges at $z = 0$. $R = 0$.

Ratio Test: $\sum_{n=0}^{\infty} c_n$. Suppose

$$\frac{c_{n+1}}{c_n} \to L \quad \text{as} \quad n \to \infty.$$

If $|L| < 1$, the series converges.
If $|L| > 1$, the series diverges.
If $|L| = 1$, the test fails.

Pf: Compare to geom series. Freshman Calc pf.

EX: $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. $\frac{c_{n+1}}{c_n} = \frac{z^{n+1}/(n+1)!}{z^n/n!} = \frac{z}{n+1} \to 0$

as $n \to \infty$ for all z. $L = 0$ in Ratio test.
$|L| < 1$. So converges for all z. $R = \infty$.

EX: $\sum_{n=0}^{\infty} \frac{z^{2n}}{3^n}$ $\frac{c_{n+1}}{c_n} = \frac{z^{2(n+1)}/3^{n+1}}{z^{2n}/3^n} = \frac{z^2}{3}$

$\left|\frac{z^2}{3}\right|$ < 1 Conv. > 1 div.

$|z| < \sqrt{3}$ Conv.
$|z| > \sqrt{3}$ Div.

$R = \sqrt{3}$

Exercise: $\{a_n\}_{n=1}^{\infty}$ be an enumeration of all points in $D_1(0)$ with rational real and imag parts. Show $\sum_0^{\infty} a_n z^n$ has R. of C. $= 1$.