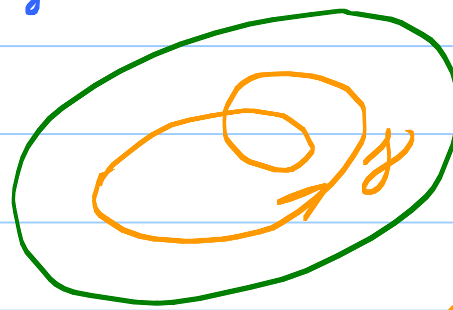
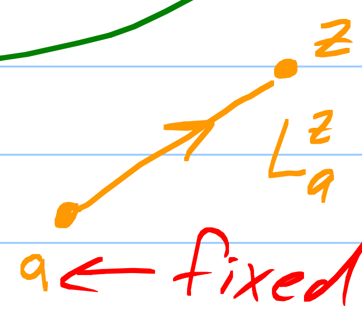
Cauchy's Theorem on a convex set:

Suppose Ω is a convex open set in Ω, and suppose f is continuous on Ω and analytic on $\Omega - \{p\}$. Then $\int_\gamma f\,dz = 0$ for any closed path in Ω.

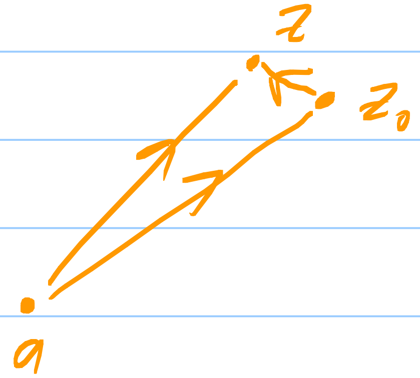


Proof: Plan: Cook up an F so that $F' = f$.

$$\left(\text{Know } \int_{\gamma_A^B} \underbrace{F'(w)}_{f}\,dw = F(B) - F(A) \right)$$



Big Idea: Define $F(z) = \int_{L_a^z} f(w)\,dw$.



Goursat: $$\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \int_{-L_a^z} \right) f\,dw = 0$$

Note: $$\int_{-L_a^z} f\,dw = -\int_{L_a^z} f\,dw.$$

See $$\int_{L_a^z} f\,dw - \int_{L_a^{z_0}} f\,dw = \int_{L_{z_0}^z} f\,dw$$

$$DQ = \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{L_{z_0}^{z}} f \, dw.$$

Let $\varepsilon > 0$. f cont on Ω, so $\exists \delta > 0$ such that $\underbrace{|f(z) - f(z_0)|}_{E(z)} < \varepsilon$ if $z \in D_\delta(z_0) \subset \Omega$.

$f(z) = f(z_0) + E(z)$ Little fact: $\int_{L_{z_0}^{z}} c \, dw =$

$$\int_{L_{z_0}^{z}} \frac{d}{dw}(cw) \, dw = (cw)\Big|_{z_0}^{z} = c(z - z_0).$$

$$DQ = \frac{1}{z - z_0} \left(\underbrace{\int_{L_{z_0}^{z}} f(z_0) \, dw}_{f(z_0)(z - z_0)} + \int_{L_{z_0}^{z}} E(w) \, dw \right)$$

$$= f(z_0) + \underbrace{\frac{1}{z - z_0} \int_{L_{z_0}^{z}} E(w) \, dw}_{\mathcal{E}(z)}$$

$$|\mathcal{E}(z)| \le \frac{1}{|z - z_0|} \underbrace{\left(\max_{tr(L_{z_0}^{z})} |E(w)| \right)}_{< \varepsilon \text{ if } |z - z_0| < \delta} \underbrace{\text{Length}(L_{z_0}^{z})}_{|z - z_0|}$$

$$\le \varepsilon \quad \text{if} \quad |z - z_0| < \delta.$$

Conclude: F is analytic on Ω and

$F' = f$ ← cont.

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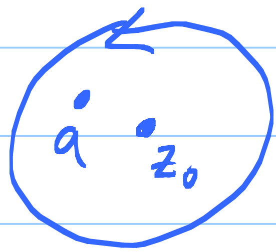
Finally, $\displaystyle\int_\gamma f\,dz = \int_\gamma F'\,dz \overset{\swarrow}{=} F(\text{END}) - F(\text{START})$

$= 0$, γ closed.

Cor: Analytic functions on a convex open set have analytic antiderivatives there.

Next stop: Cauchy Integral Formula on a disc.

Lemma:



$C_r(z_0)$: $z(t) = z_0 + re^{it}$

$0 \le t \le 2\pi$

$$\int_{C_r(z_0)} \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & \text{if } a \in D_r(z_0) \\ 0 & \text{if } |a - z_0| > r \end{cases}$$

Pf: 0 case is just Cauchy Thm on convex $D_\rho(z_0)$ where $r < \rho < |a - z_0|$.

Case $a = z_0$: $\displaystyle\int_{C_r(z_0)} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{\underbrace{(z_0+re^{it})}_{z(t)} - z_0} \underbrace{ire^{it}}_{z'(t)}\,dt$

$\displaystyle = i\int_0^{2\pi} 1\,dt = 2\pi i$ ✓

Case $a \in D_r(z_0)$: Define $H(w) = \int_{C_r(z_0)} \frac{1}{z-w}\, dz$.

HWK 2, #1 shows $H'(w) = \int_{C_r(z_0)} \frac{1}{(z-w)^2}\, dz =$

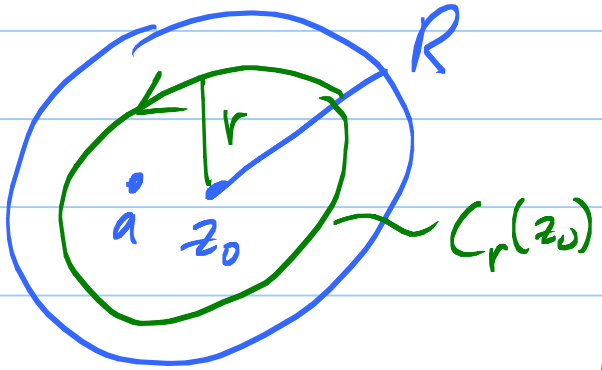
$$= \int_{C_r(z_0)} \frac{d}{dz}\left[\frac{-1}{(z-w)}\right] dz \equiv 0.$$

↑ Fund. Thm. Calc 2

So $H(w) \equiv$ const. But $H(z_0) = 2\pi i$,

So $H(w) \equiv 2\pi i$.

Cauchy Integral Formula on a disc:



f analytic on $D_R(z_0)$,

$0 < r < R$,

$a \in D_r(z_0)$.

$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz$$

Pf: $D_R(z_0)$ is convex, open. (Think $p=a$)

Define $$G(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

G is cont. on $D_R(z_0)$, analytic on $D_R(z_0) - \{a\}$.

Cauchy on convex $\Rightarrow \displaystyle\int_{C_r(z_0)} G(z)\,dz = 0$

$$\int_{C_r(z_0)} \frac{f(z) - f(a)}{z-a}\,dz = 0$$

$$\int_{C_r(z_0)} \frac{f(z)}{z-a}\,dz - f(a)\underbrace{\int_{C_r(z_0)} \frac{1}{z-a}\,dz}_{=2\pi i \text{ by Lemma}} = 0 \quad \checkmark$$

Big consequence: f analytic $\Rightarrow$ f' analytic.

Why: f = C.I.F.

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \quad \leftarrow \text{Hwk 2, \#1}$$

$$f''(a) = \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Cauchy Formulas

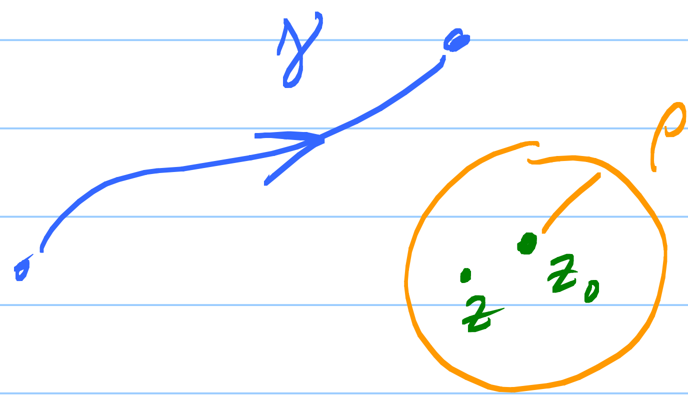
HWK 2, #1 :

$$f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw$$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{z-z_0}\left[\int_\gamma \varphi(w)\left(\frac{1}{w-z} - \frac{1}{w-z_0}\right)dw\right]$$

$$= \int_\gamma \frac{\varphi(w)}{(w-z)(w-z_0)}\, dw \xrightarrow{\text{expect}} \underbrace{\int_\gamma \frac{\varphi(w)}{(w-z_0)^2}\, dw}_{I}$$

$$\left| DQ - I \right| = |z-z_0| \left| \int_\gamma \frac{\varphi(w)}{(w-z)(w-z_0)^2}\, dw \right|$$



$$d = \operatorname{dist}(z_0, \gamma) = \operatorname{Inf}\{ |z(t) - z_0| : a \le t \le b \}$$

Take $\rho = \frac{d}{2}$,