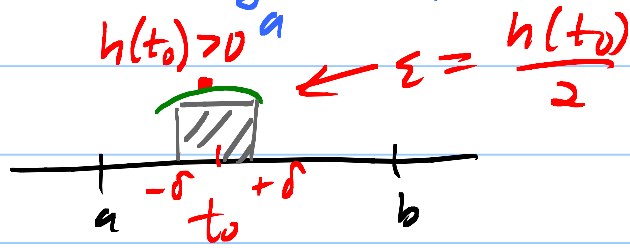
Calculus Lemma: If $h : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $h(t) \geq 0$ for all t, then $\int_a^b h(t)\,dt = 0 \implies h \equiv 0$.

Pf:



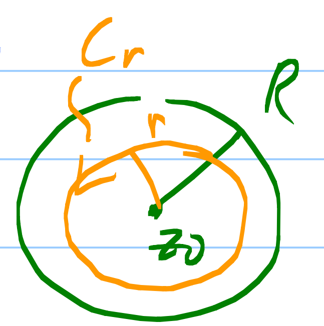
$\int > 2\delta\varepsilon$ ↯

Lemma: If $h : [a,b] \to \mathbb{R}$ is continuous and $0 \leq h(t) \leq M$ for all t, then

$$\int_a^b h(t)\,dt = M(b-a) \implies h \equiv M.$$

Pf: Replace h by $M-h$ above.

Averaging Property for analytic fcns.



Suppose f analytic on $D_R(z_0)$.

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$$= \frac{1}{2\pi r}\int_0^{2\pi} f(z_0+re^{it})\,\underbrace{r\,dt}_{d\theta} = \text{Ave over } C_r \text{ w.r.t. arc length.}$$

length of C_r

Fact: Ave Property $\implies$ Max Principle.

Pf of Ave Prop:

$C_r : z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z_0}\, dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + re^{it}) - z_0}\, ire^{it}\, dt$$

= average ✓

Maximum Modulus Principle #1: Suppose f is analytic on a domain Ω. If $|f|$ attains a local maximum at a point in Ω, then f is constant on Ω.

Note: z_0 local max of $|f|$ means:

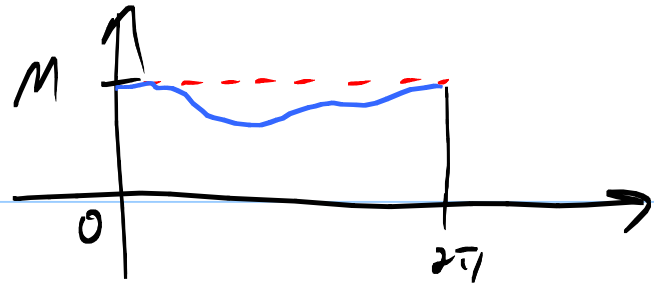
$\exists\, r > 0$ such that $D_R(z_0) \subset \Omega$ and

$|f(z)| \le |f(z_0)|$ for all $z \in D_R(z_0)$.

Pf of Max Princ: Suppose $z_0 \in \Omega$ is a local max, $D_R(z_0)$ as above. Then r with $0 < r < R$,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt. \quad \text{So}$$

$$\underbrace{|f(z_0)|}_{M} \le \frac{1}{2\pi} \int_0^{2\pi} \underbrace{|f(z_0 + re^{it})|}_{h(t) \le M = |f(z_0)|}\, dt \le M$$



Calculus Lemma $\Rightarrow |f(z_0+re^{it})| \equiv M$ for $0 \le t \le 2\pi$.

r arbitrary $\Rightarrow |f| \equiv M$ on $D_R(z_0)$.

HWK 3, prob. 4 $\Rightarrow f$ const on $D_R(z_0)$.

$\Rightarrow f$ const on Ω by Identity Thm.

Maximum Principle #2: Suppose Ω is a bounded domain, f continuous on $\overline{\Omega}$, f analytic on Ω. Then $|f|$ attains its maximum value at a point on the boundary of Ω.

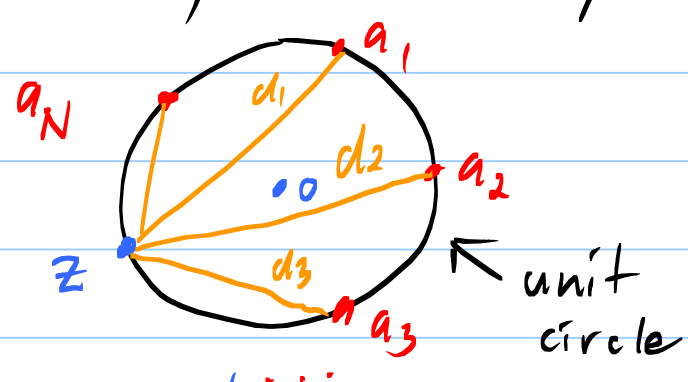
Pf: $\overline{\Omega}$ is closed and bounded, and therefore compact. A continuous fcn on a compact set attains its max. Suppose $|f|$ attains its max at $w_0 \in \overline{\Omega}$.

Case 1: w_0 a boundary point. done!

Case 2: $w_0 \in \Omega$. Max Princ #1 $\Rightarrow |f|$ const on Ω. Claim: This implies $|f|$ = same constant on boundary.

Because $z_0 \in$ boundary means $\exists$ seq $z_n \in \Omega$ with $z_n \to z_0$ and seq $w_n \in \Omega^c$ with $w_n \to z_0$. So $f(z_n) \to f(z_0)$ = same constant by continuity.

EX:



Fact: $\exists$ z in unit circle where $\prod_{k=1}^{N} d_k = 1$.

$$f(z) = \prod_{k=1}^{N} (z - a_k) \leftarrow \text{not const., analytic}$$

Max Princ #2 $\Rightarrow$ $|f|$ attains max on $C_1(0)$.

Hmmm. $f(0) = \prod_{k=1}^{N} a_k$. So $|f(0)| = 1$.

$|f(a_k)| = 0$. Aha!

Max Princ #2 $\exists$ θ_0 where $|f(e^{i\theta_0})| = M > 1$.

$e^{i\theta_k} = a_k$. $\exists$ α_k between θ_0 and θ_k where $|f(e^{i\alpha_k})| = 1$ by I.V.T.

Remark: z^n on $D_1(0)$ shows no Minimum Principle. Ah, but there is!

Thm: f continuous on $\overline{\Omega}$ and f non-vanishing on Ω and f analytic on Ω. Then Min Princ holds.

Pf: Apply max princ to $1/f$.

EX: f continuous on $\overline{D_1(0)}$, analytic on $D_1(0)$, and $|f| \equiv 1$ on $C_1(0)$, and f non-vanishing on $D_1(0)$. Then $f(z) \equiv \lambda$, λ a constant of modulus 1.

Pf: Max Princ for $f \Rightarrow |f| \le 1$.

Max Princ for $1/f \Rightarrow \left|\frac{1}{f}\right| \le 1$.

So $|f| \equiv 1$ on $D_1(0)$. $\Rightarrow f \equiv$ const.

Theorem: Suppose u is a C^2-smooth $\mathbb{R}$-valued harmonic fcn on $D_R(z_0)$. Then u has a harmonic conjugate v on $D_R(z_0)$ [i.e., $u+iv$ is analytic]. Consequently u is C^∞-smooth!

Remark: f analytic $\Rightarrow f'$ analytic $\Rightarrow$ ----

$\Rightarrow u, v$ C^∞ smooth via C-R eqns