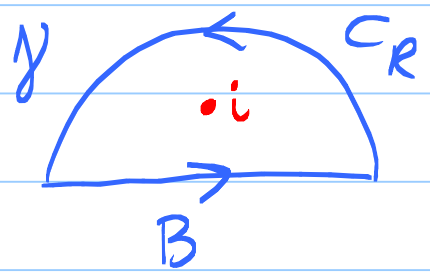
Fourier Transform:

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

EX: $I(s) = \displaystyle\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx \quad , \quad s>0.$

$$g(z) = \frac{e^{isz}}{z^2+1} \qquad \left|e^{is(x+iy)}\right| = \left|e^{isx}e^{-sy}\right| = e^{-sy}$$

≤ 1 when $y \geq 0$



$$\int_B g(z)\,dz = \int_{-R}^{R} \frac{e^{isx}}{x^2+1}\,dx$$

≤ 1 on C_R

$$\left|\int_{C_R} g(z)\,dz\right| \leq \left(\max_{z\in C_R} \frac{|e^{isz}|}{|z^2+1|}\right)\pi R$$

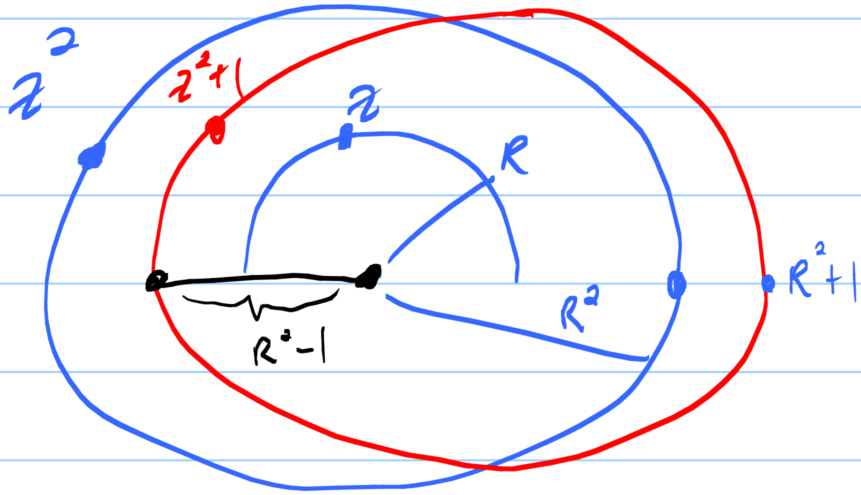
$$\leq \frac{1}{R^2-1}\cdot \pi R \quad \text{if } R>1$$

$$\longrightarrow 0 \text{ as } R\to\infty.$$

Residue Thm: $\displaystyle\int_\gamma = \int_B + \int_{C_R} = 2\pi i \,\mathrm{Res}_i \frac{e^{isz}}{z^2+1}$

$$I(s) + 0 = 2\pi i \cdot \frac{e^{is(i)}}{2i}$$

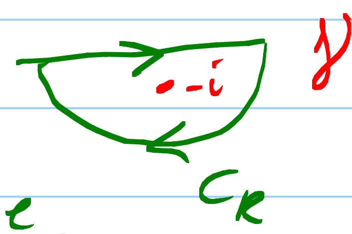
$$= \pi e^{-s}$$



$$\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}$$

$f(a)=0,\ f'(a)\neq 0$

Remark: What if $s<0$?



$|e^{isz}| \le 1$ in lower half plane.

$$\int_\gamma = -2\pi i \,\mathrm{Res}_{-i} \frac{e^{isz}}{z^2+1}$$

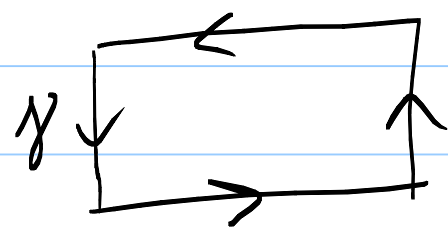
clockwise!

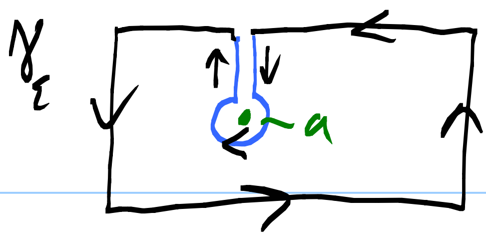
Me:

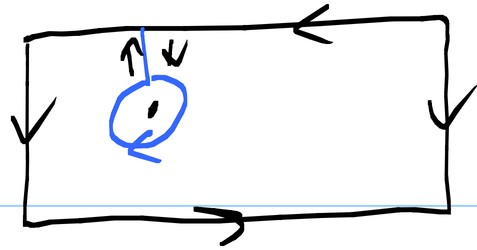
$$\int_\gamma \frac{1}{z-w}\,dz = 2\pi i$$

w inside γ.

Stein: Keyhole argument.







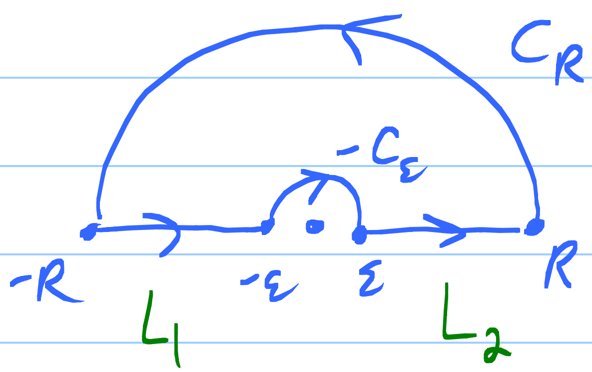
Cauchy on Toy: $0=\int_{\gamma_\varepsilon} \frac{1}{z-a}\,dz$

$$\int_{Box} + \left(-\underbrace{\int_{C_\varepsilon} \frac{1}{z-a}\,dz}_{2\pi i}\right) = 0$$

So $\int_{Box} \frac{1}{z-a}\,dz = 2\pi i$

Famous calculation: $I = \int_0^\infty \frac{\sin t}{t}\,dt$

$f(z) = \frac{e^{iz}}{z}$ $\quad f(t) = \frac{e^{it}}{t} = \frac{\cos t}{t} + i\,\frac{\sin t}{t}, \; t \in \mathbb{R}.$



$$\left(\int_{L_1} + \int_{L_2}\right) \frac{\cos t}{t}\,dt = 0$$

odd

$$\left(\int_{L_1} + \int_{L_2}\right) \frac{\sin t}{t}\,dt = 2\int_\varepsilon^R \frac{\sin t}{t}\,dt$$

even

So $\left(\int_{L_1} + \int_{L_2}\right) \frac{e^{iz}}{z}\,dz = 2i\int_\varepsilon^R \frac{\sin t}{t}\,dt$ ← want

On $C_{-\varepsilon}$: $\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{1}{2!}(iz)^2 + \cdots}{z}$

$$= \frac{1}{z} + \Big[\underbrace{i + \frac{i^2}{2!} z + \cdots}_{H(z),\ \text{entire!}} \Big]$$

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz = \underbrace{\int_{-C_\varepsilon} \frac{1}{z}\, dz} + \underbrace{\int_{-C_\varepsilon} H(z)\, dz}$$

$C_\varepsilon : \varepsilon e^{it}$

$0 \le t \le \pi$

$$= -\int_{C_\varepsilon} \frac{1}{z}\, dz$$

$$= -\int_0^{\pi} \frac{1}{(\varepsilon e^{it})} \underbrace{i\varepsilon e^{it}\, dt}_{dz}$$

$$= -i\pi$$

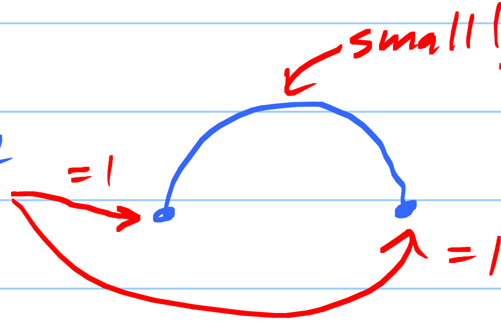
$$\left| \int_{-C_\varepsilon} H\, dz \right| \le \Big(\max_{|\overline{D_1(0)}|} |H|\Big)\, \pi\varepsilon$$

$$\to 0 \text{ as } \varepsilon \to 0$$

On C_R :

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \le \max_{C_R} \left| \frac{e^{iz}}{z} \right| \cdot \pi R$$

$$\left| e^{i(x+iy)} \right| = \left| e^{ix} e^{-y} \right| = e^{-y}$$



$$\le \frac{1}{R} \cdot \pi R \quad \text{ouch!}$$

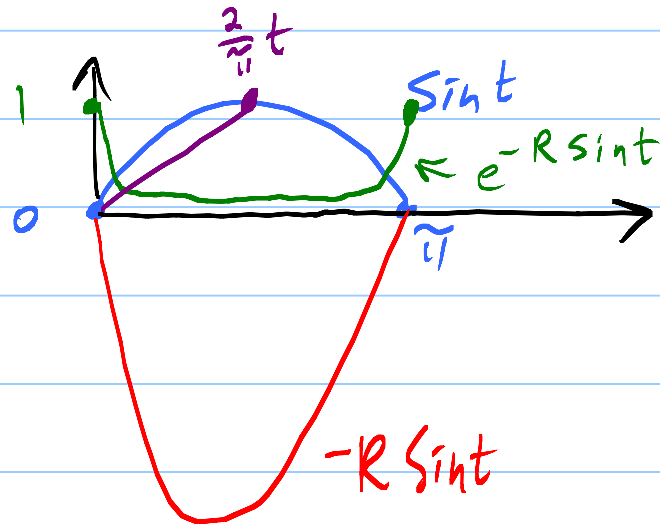
Need to be fancier. Jordan's Lemma.

$C_R : \ z(t) = Re^{it} = R\cos t + iR\sin t, \quad 0 \le t \le \pi.$

$$\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi}\frac{e^{i(R\cos t + iR\sin t)}}{Re^{it}}\,\underbrace{iRe^{it}\,dt}_{dz}\right|$$

$$\leq \int_0^{\pi} \left| \sim\sim \right| dt = \int_0^{\pi} e^{-R\sin t}\,dt$$

$\parallel \leftarrow$ symmetry



$$2\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$0 \leq \frac{2}{\pi}t \leq \sin t, \quad [0, \tfrac{\pi}{2}]$$

$$-R\sin t \leq -\frac{2R}{\pi}t \leq 0$$

$$e^{-R\sin t} \leq e^{-\frac{2R}{\pi}t} \leq 1$$

So $$\int_0^{\pi/2} e^{-R\sin t}\,dt \leq \int_0^{\pi/2} e^{-(2R/\pi)t}\,dt$$

$$= \left[-\frac{\pi}{2R}e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2R}\left(1 - e^{-R}\right) < \frac{\pi}{2R}$$

$$\longrightarrow 0 \text{ as } R \to \infty.$$

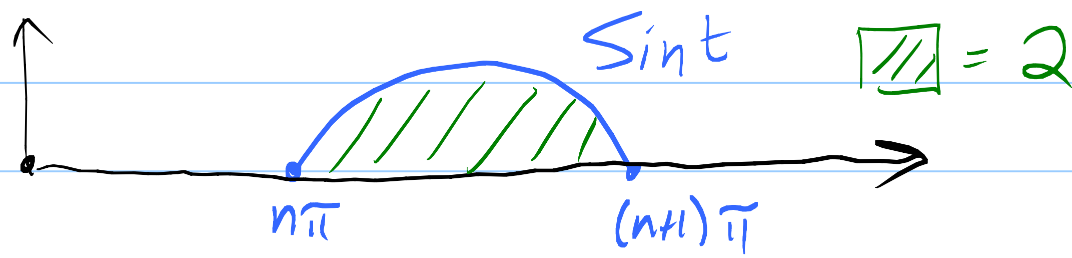
Cauchy on Toys $\displaystyle\int_\gamma \frac{e^{iz}}{z}\,dz = 0$

$$\underbrace{\int_{L_1} + \int_{L_2}} + \int_{-C_\varepsilon} + \int_{C_R} = 0$$

$$\varepsilon \to 0:\ 2i\int_0^R \frac{\sin t}{t}\,dt + (-i\pi) + \int_{C_R} = 0$$

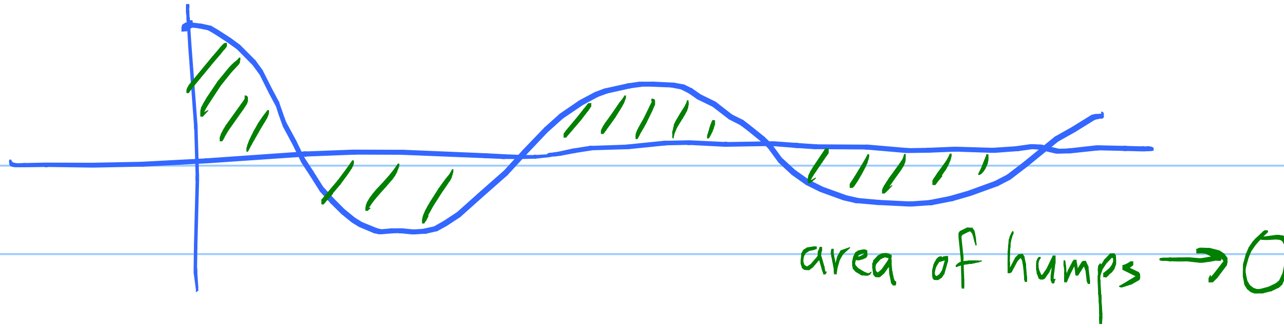
$$R \to \infty:\ 2i\int_0^\infty \frac{\sin t}{t}\,dt + (-i\pi) + 0 = 0$$

$I = \frac{\pi}{2}$! $\displaystyle\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = \pi$



$$\frac{|\sin t|}{(n+1)\pi} \le \left|\frac{\sin t}{t}\right| \le \frac{|\sin t|}{n\pi} \longleftarrow \int_{n\pi}^{(n+1)\pi}$$

$$\frac{2}{(n+1)\pi} \le \left|\int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\,dt\right| \le \frac{2}{n\pi}$$



Alternating series test shows $\int$ converges.

<u>But</u> $\int_0^\infty \left|\frac{\sin t}{t}\right| dt \sim \sum_2^\infty \frac{1}{n} = \infty$. Conditional convergence. Not <u>absolute</u>.