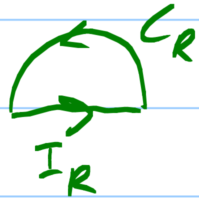
Theorem: P, Q polynomials, Q has no zeroes on $\mathbb{R}$, $\deg Q \geq \deg P + 1$.

Then
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in UHP}\}$.

Pf:



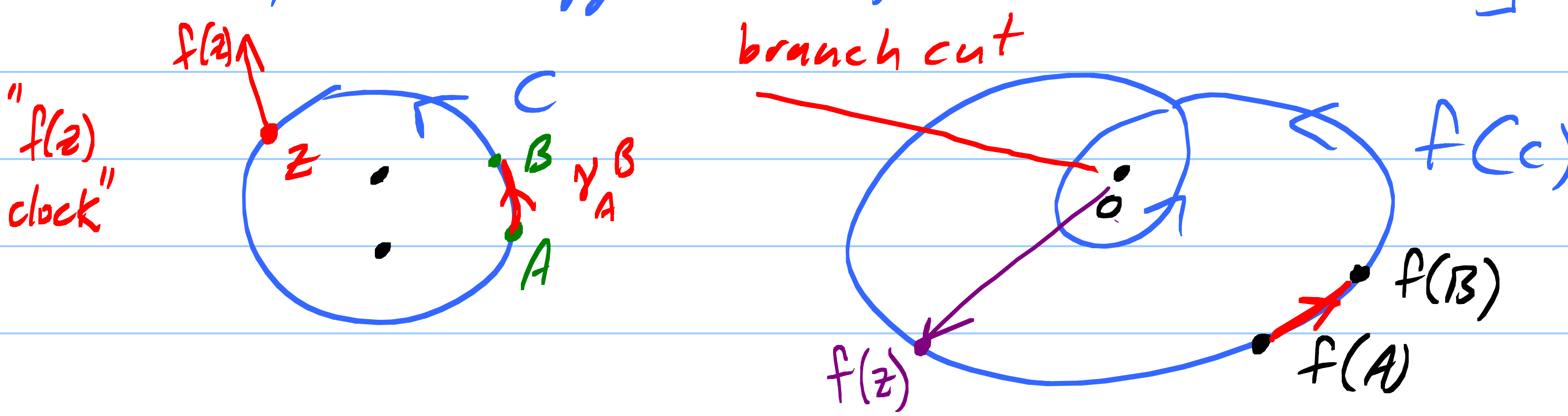
Show $\int_{C_R} \frac{P}{Q} e^{iz}\, dz \to 0$.

Key points: 1) Basic poly est $\Rightarrow \left| \frac{P(z)}{Q(z)} \right| \leq \frac{C}{|z|}$ if $|z| > R_0$.

2) Jordan's Lemma: $\left| \int_{C_R} \right| \leq$ something $\to 0$.

Why "Argument Principle": $\frac{1}{2\pi i} \int_C \frac{f'}{f}\, dz = \left(\begin{array}{c} \#\text{ zeroes} \\ \text{of } f \\ \text{inside } C \end{array} \right)$

[f analytic on bigger disc, no zeroes on C.]



$$\frac{d}{dz}(\log z) = \frac{1}{z}$$

True: $e^{\log z} = z$

Chain rule $\underbrace{e^{\log z}}_{z} \frac{d \log z}{dz} = 1$

[Not always true: $\log(e^z) = z$ plus possible multiple of $2\pi i$]

Note: $\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$ ← Aha!

$$\int_{\gamma_A^B} \frac{f'}{f}\, dz = \int_{\gamma_A^B} \frac{d}{dz}(\log f)\, dz = \log f(B) - \log f(A)$$

$$= \left(\operatorname{Ln}|f(B)| + i \arg f(B)\right) - \left(\operatorname{Ln}|f(A)| + i \arg f(A)\right)$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i \Delta_{\gamma_A^B} \arg f$$ ← indep of branch!

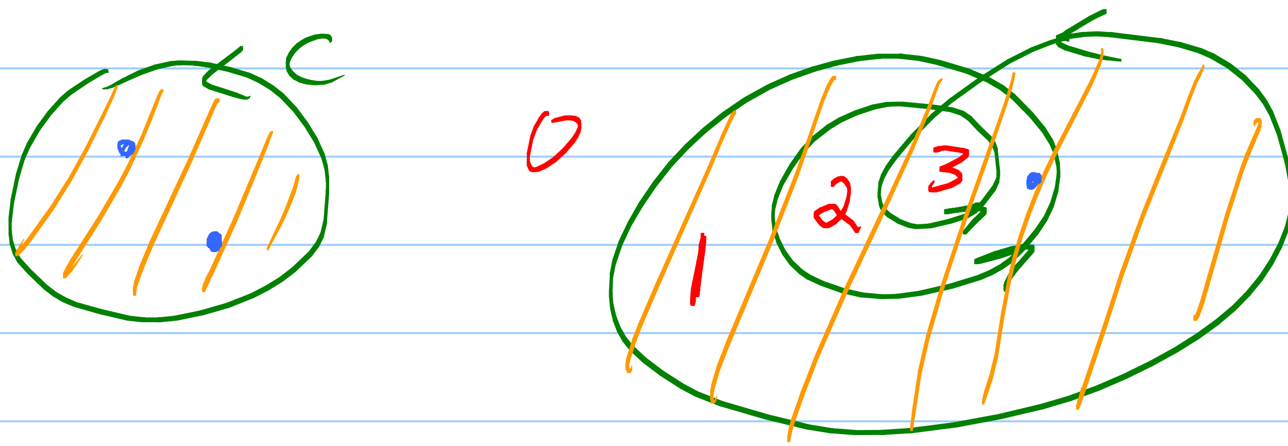
Add up segments around C. Real parts pairwise cancel. Get

$$\int_C \frac{f'}{f}\, dz = i \Delta_C \arg f$$

$$\frac{1}{2\pi i} \int_C \frac{f'}{f}\, dz = \left(\begin{array}{c}\#\text{ times } f \\ \text{winds around } 0\end{array}\right)$$ ← $F(z) = f(z) - w$

Note: $\displaystyle \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz = \begin{pmatrix}\#\text{ times} \\ f(z)=w \\ \text{inside } C\end{pmatrix}$

$= \begin{pmatrix}\#\text{ times } f(z) \text{ winds around} \\ w \text{ as } z \text{ goes around } C\end{pmatrix}$



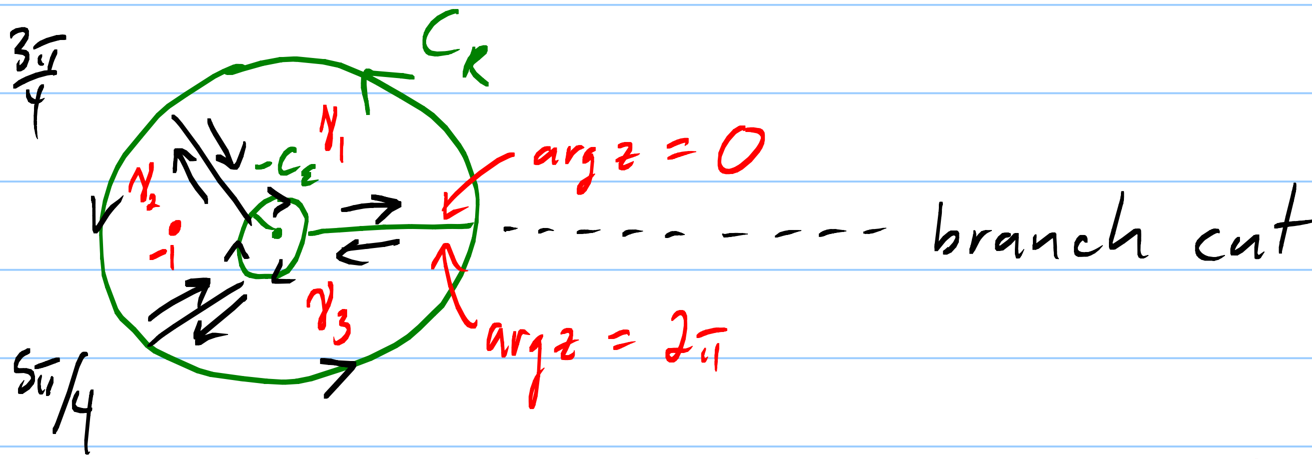
Important fact: $\displaystyle H(w) = \int_C \frac{f'(z)}{f(z)-w}\,dz$

$= \begin{pmatrix}\#\text{ times } f(z)=w \\ \text{inside } C\end{pmatrix}$

OMT pf showed $H'(w) \equiv 0$. So $H(w)$ is constant on connected components of $\mathbb{C} - f(C)$.

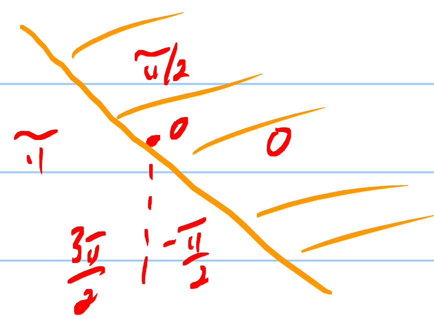
EX: $\displaystyle I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx, \quad 0<\alpha<1$

$$x^{-\alpha} = e^{-\alpha \ln x}, \qquad z^{-\alpha} = e^{-\alpha \log z}$$

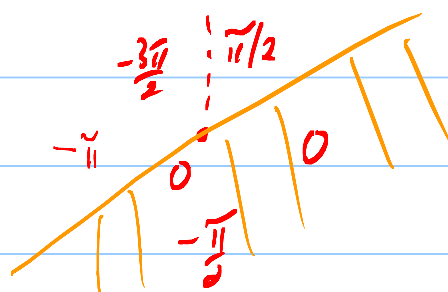


$$\int_{\gamma_1} \underbrace{\frac{e^{-\alpha \log z}}{z+1}}_{F(z)} \, dz = 0$$

Cauchy Thm on convex



Same for γ_3:



$$\int_{\gamma_2} F(z)\, dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1} = 2\pi i \, \frac{e^{-\alpha \log(-1)}}{1}$$

$$= 2\pi i \; e^{-\alpha\left(\ln|-1| + i \overbrace{\arg(-1)}^{\pi}\right)} = 2\pi i \, e^{-i\alpha\pi}$$

Next: $\left| e^{-\alpha \log z} \right| = \left| e^{-\alpha(\ln|z| + i \arg z)} \right.$

$$= e^{-\alpha \ln|z|} \underbrace{\left|e^{-i\alpha \arg z}\right|}_{1} = |z|^{-\alpha}$$

Aha! $|z^p| = |z|^p$

Use this estimate to show $\left|\int_{C_R} F(z)\,dz\right| \to 0$ as $R \to \infty$ $(\alpha > 0)$. and $\left|\int_{-C_\varepsilon} F(z)\,dz\right| \to 0$ as $\varepsilon \to 0$ $(\alpha < 1)$.

$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) F\,dz = 2\pi i\, e^{-i\alpha\pi}$$

$$= \left(\int_{C_R} + \int_{C_\varepsilon} + \int_{\text{Top of } \mathbb{R}} + \underbrace{\int_{\text{Bottom of } \mathbb{R}}}\right) F\,dz$$

$$0 + 0 + \int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\,dt + (?) =$$

$(?)$: $\varepsilon \xleftarrow{-L} R$ $\qquad L: z(t) = t, \quad \varepsilon \le t \le R$

$$\int_{-L} \frac{e^{-\alpha \log z}}{z+1}\,dz = -\int_L = -\int_\varepsilon^R \frac{e^{-\alpha(\ln t + i2\pi)}}{t+1}\,dt$$

$$= -e^{-i\alpha 2\pi} \underbrace{\int_{\varepsilon}^{R} \frac{t^{-\alpha}}{t+1}\, dt}_{\text{want this!}}$$

Step 1: Let $\varepsilon \to 0$. Step 2: Let $R \to \infty$.

$$\left(1 - e^{-i\alpha 2\pi}\right) I = 2\pi i\, e^{-i\alpha\pi}$$

Solve for I. Get $\dfrac{\pi}{\sin \alpha\pi}$.

$$\left| \int_{-C_\varepsilon} \frac{e^{-\alpha \log z}}{z+1}\, dz \right| \le \underset{C_\varepsilon}{\mathrm{Max}} \frac{\varepsilon^{-\alpha}}{1-\varepsilon} \cdot (2\pi\varepsilon)$$

$$|z+1| = |z-(-1)| \ge \big| \underbrace{|z|}_{\varepsilon} - |-1| \big| = 1-\varepsilon$$

if $\varepsilon < 1$.

$$\left| \int_{C_R} dz \right| \le \frac{R^{-\alpha}}{R-1} (2\pi R)$$

$\alpha < 1$

$0 < \alpha$

Hwk 6 due Wed after break.