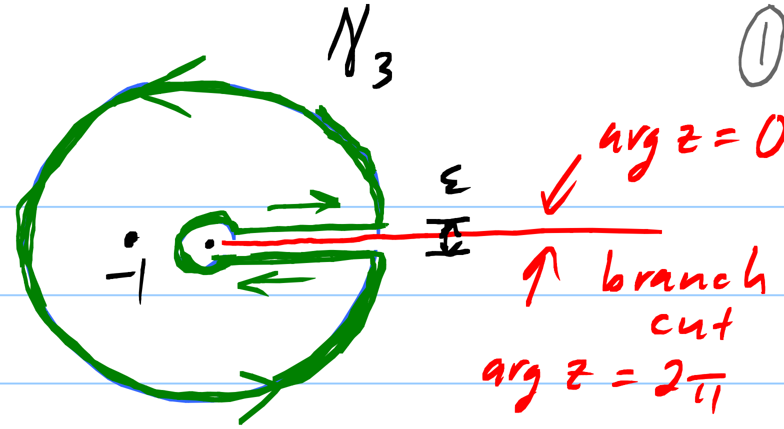


Read Stein Chapter 3 & Linear Fractional Transf.

$$\text{Stein}: \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{e^{-\alpha \log z}}{z+1}\, dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1}$$

Trig Integrals $\int_0^{2\pi}$: $C: z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Note that $dz = z'(\theta)\, d\theta = i e^{i\theta}\, d\theta$

$$\text{EX}: \int_0^{2\pi} \cos^6\theta\, d\theta = \int_0^{2\pi} \left[\frac{z(\theta) + \frac{1}{z(\theta)}}{2}\right]^6 \frac{\overset{dz}{i e^{i\theta} d\theta}}{i \underbrace{e^{i\theta}}_{z(\theta)}}$$

$$= \int_C \left[\frac{z + \frac{1}{z}}{2}\right]^6 \frac{1}{iz}\, dz$$

$$= \frac{1}{i 2^6} \int_C \left(z^6 + 6z^4 + 15z^2 + \overset{\mathrm{Res}_0}{20} + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6} \right) \frac{1}{z}\, dz$$

$= \frac{1}{i2^6} \cdot 2\pi i \cdot 20$ by Residue Thm.

$= \frac{5\pi}{8}$.

General Fact: $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

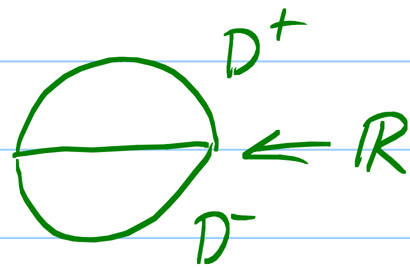
rational

$= \int_C R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{1}{iz}\, dz$

= use Residue Thm.

Schwarz Reflection Principle:

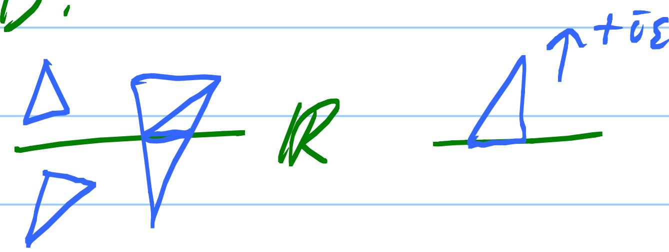
Preliminary fact: D



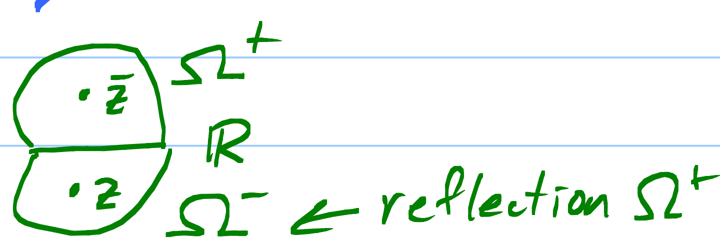
f continuous on D, analytic on $D^{\pm}$,

then f is analytic on D.

Pf: Morrera's Thm



Schwarz Reflection: Ω



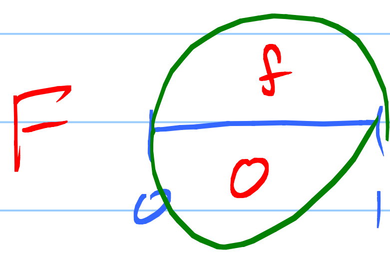
Suppose f is analytic on Ω^+ and continuous up to $\mathbb{R}$ and real there. Define

$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \cup \mathbb{R} \leftarrow \text{analytic } \Omega^+ \\ \overline{f(\bar{z})} & z \in \Omega^- \leftarrow \text{analytic on } \Omega^- \text{ (hwk)} \end{cases}$$

Then F is analytic on Ω.

Pf: $\overline{f(\bar{z})}$ when $z = x \in \mathbb{R}$ $= \overline{f(x)} = f(x)$ because $f(x)$ is real when x is. So F is continuous on Ω. Prel. Fact $\Rightarrow$ Thm.

Application: Suppose f is analytic on UHP and continuous up to $(0,1)$ on $\mathbb{R}$. If f vanishes on $(0,1)$, then $f \equiv 0$ on the UHP.

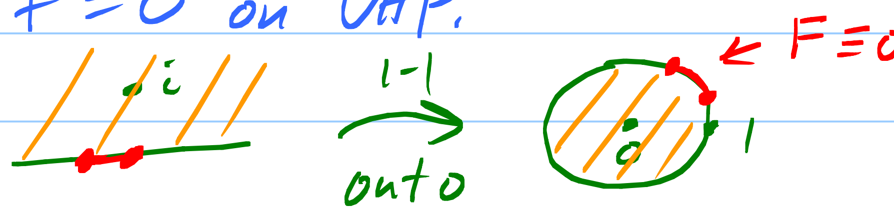
Pf: 

$\leftarrow$ Prel. Fact $\Rightarrow$ F analytic

Iden. Thm $\Rightarrow$ $F \equiv 0$.

So $f \equiv 0$ on D.

Iden. Thm. $\Rightarrow$ $f \equiv 0$ on UHP.

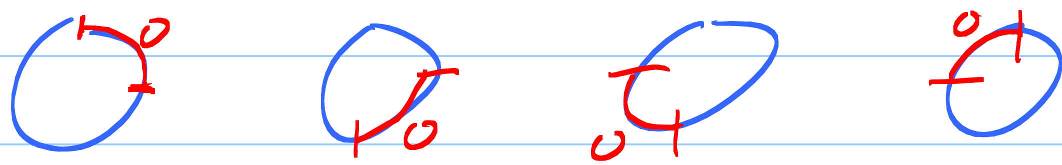
EX: $\dfrac{z-i}{z+i}$ 

Fact: f analytic on $D_r(z_0)$ and continuous up to the boundary. If f vanishes on an open subset of the boundary, then $f \equiv 0$.

Old Qual Question: Suppose f is analytic on $D_1(0)$ and f continuous up to the boundary. If $f(e^{i\theta}) = 0$ for $0 \le \theta \le \frac{\pi}{2}$, then $f \equiv 0$ on $D_1(0)$.

Pf: Schwarz works. Or define

$$F(z) = f(z)\, f(e^{i\pi/2} z)\, f(e^{i\pi} z)\, f(e^{i3\pi/2} z)$$



$F \equiv 0$ on boundary, analytic inside.

Max Princ. $\Rightarrow$ $F \equiv 0$ on $D_1(0)$.

Take $z = 1/n$, $n = 2, 3, \ldots$. Get infinite subseq where one of four $= 0$. Iden. Thm. $\Rightarrow$ one of them $\equiv 0$. $\Rightarrow f \equiv 0$.

Linear Fractional Transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d} \quad \text{where} \quad ad-bc \neq 0.$$

Remark: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$, then if row 1 $=0$, $L(z) \equiv 0$. If row 2 $=0$, then L undefined. Otherwise row 2 is a const times row 1 and $L(z) \equiv \text{const}$.

$$L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A \qquad L^{-1} \sim A^{-1}$$

$$L_1 \circ L_2 \sim A_1 A_2$$

Fact: L^{-1} is a LFT.

$$w = \frac{az+b}{cz+d} \leftarrow \text{solve for } z \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$\text{get} \quad z = \frac{dw-b}{-cw+a} \qquad \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$$

note: multiplying by $\frac{1}{\det(A)}$ doesn't change this.

Fact: $w = L(z) = \dfrac{az+b}{cz+d}$, $z = L^{-1}(w) = \dfrac{dw-b}{-cw+a}$.