LFTs $\quad L(z) = \dfrac{az+b}{cz+d}, \quad ad-bc \neq 0.$

Lemma: LFTs are generated by

1) $z \mapsto rz, \quad r \in \mathbb{R}^+$ (Stretch)
2) $z \mapsto e^{i\theta} z$ (Rotation)
3) $z \mapsto z + b$ (Translation)
4) $z \mapsto 1/z$ (Inversion)

Pf: $c=0$ case: $L(z) = \underset{\uparrow\, re^{i\theta}}{A}z + B$ 1,2,3 generate

$c \neq 0$ case: $$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\overbrace{\left(b - \frac{ad}{c}\right)}^{\leftarrow Re^{i\psi}}}{\underset{\uparrow\, re^{i\theta}}{c}z+d}$$

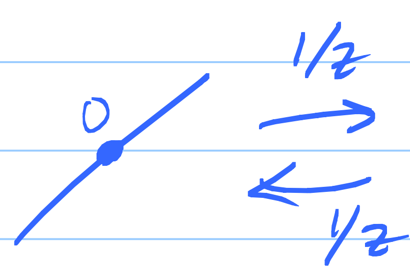
(long division).

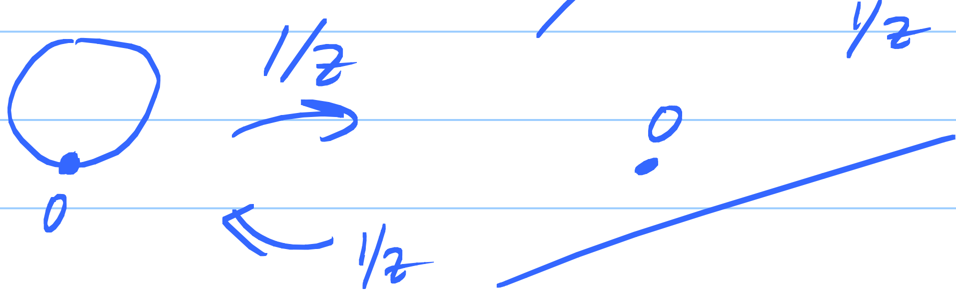
$\exists$ compositions of 1-4!

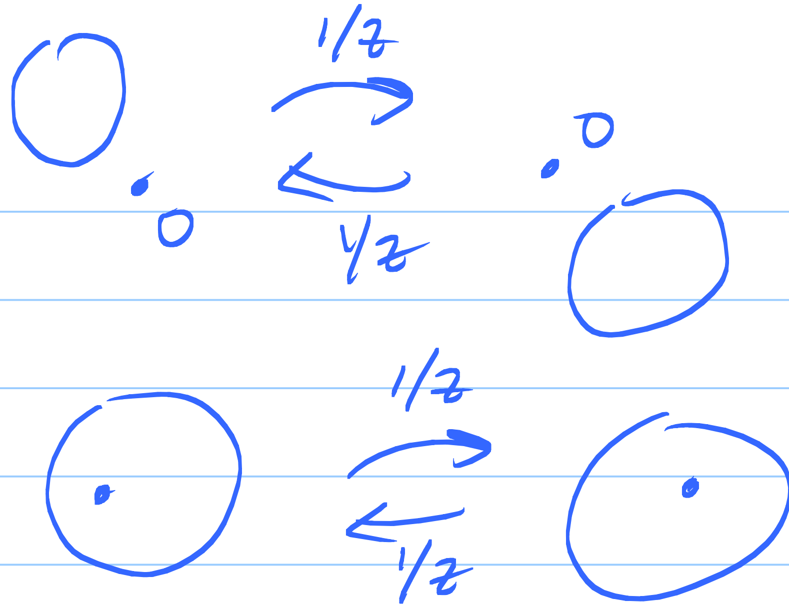
Cor: LFTs take $\{\text{lines, circles}\} \circlearrowleft$

Pf: 1,2,3 takes lines to lines, circles to circles.

4 can mix them up:







Fact: LFTs $\mathbb{C} \cup \{\infty\} \circlearrowleft$ $\overset{1-1}{\text{onto}}$

LFTs $= \text{Aut}(\hat{\mathbb{C}})$

Note: $c \neq 0$; $\quad L(z) = \dfrac{az+b}{cz+d}$

$\lim_{z \to \infty} L(z) = \frac{a}{c}$. Let $L(\infty) = \frac{a}{c}$.

$\lim_{z \to -\frac{d}{c}} L(z) = \infty$. Let $L(-\frac{d}{c}) = \infty$.

$c = 0$: $L(z) = Az + B$ Let $L(\infty) = \infty$.

Remark: On Riemann Sphere, {lines, circles} in $\mathbb{C}$ correspond to slices of S' by planes, i.e., circles on sphere.

③

Lemma: An LFT that fixes 3 pts must be the identity.

Pf: $L(z_j) = z_j \qquad j = 1, 2, 3.$

$$\frac{az_j + b}{cz_j + d} = z_j \qquad \boxed{cz_j^2 + (d-a)z_j - b = 0}$$

Quadratic with 3 roots!

Must have $c=0$, $d-a=0$, $b=0$.

$$L(z) = \frac{az}{d} = z \checkmark$$

Cor: If L_1 and L_2 agree at 3 pts, they $L_1 \equiv L_2$.

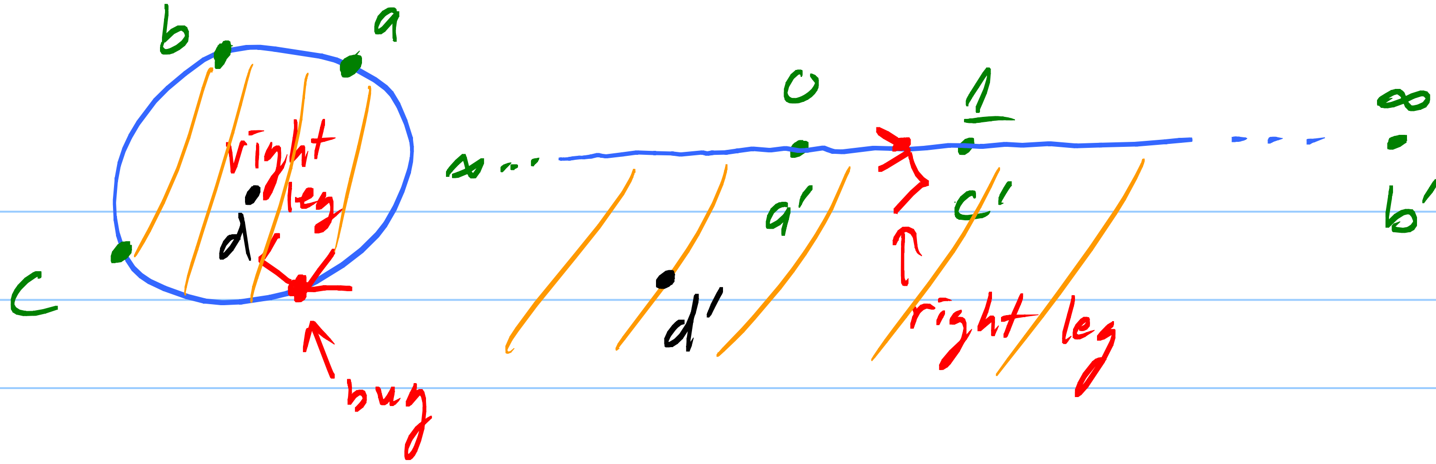
Notes: 3 points in $\mathbb{C}$ determine a circle!

If a {line, circle} contains ∞, then it is a line.

Useful LFTs: **EX:** $\dfrac{z-a}{z-b}$ $\qquad a \mapsto 0$, $b \mapsto \infty$

Hmmm. Can a 3rd point c go to 1?

EX: $\dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}$ $\qquad a \mapsto 0$, $b \mapsto \infty$, $c \mapsto 1$



Conformality implies: inside maps to lower HP.

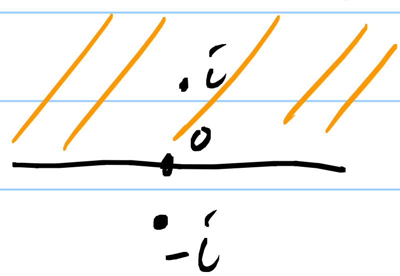
Prob: Find LFT mapping $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$.

Sol^n: $L_1(z) = \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{z - z_1}{z - z_2}$ $\quad z_1 \mapsto 0,\ z_2 \mapsto \infty,\ z_3 \mapsto 1$

$L_2(w) = \dfrac{w_3 - w_2}{w_3 - w_1} \cdot \dfrac{w - w_1}{w - w_2}$ $\quad w_1 \mapsto 0,\ w_2 \mapsto \infty,\ w_3 \mapsto 1$

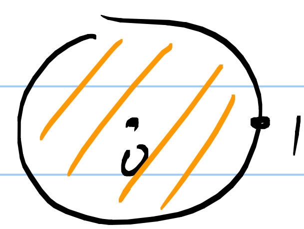
Ans: $L = L_2^{-1} \circ L_1$

EX:

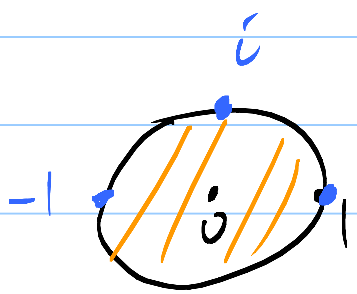


$\xrightarrow{\frac{z-i}{z+i}}$

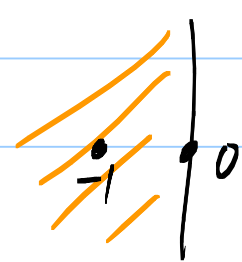
$\xleftarrow{\qquad}$



$$\frac{iz + i}{-z + 1} = -i\,\frac{z+1}{z-1}$$



$\xrightarrow{\frac{z-1}{z+1}}$



Prob: Find a conformal mapping of

onto the UHP.

Idea: Blast a corner to ∞ via a LFT.

$$L(z) = \frac{z+1}{z-1}$$

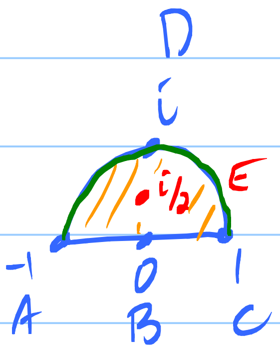
$L(0) = -1$

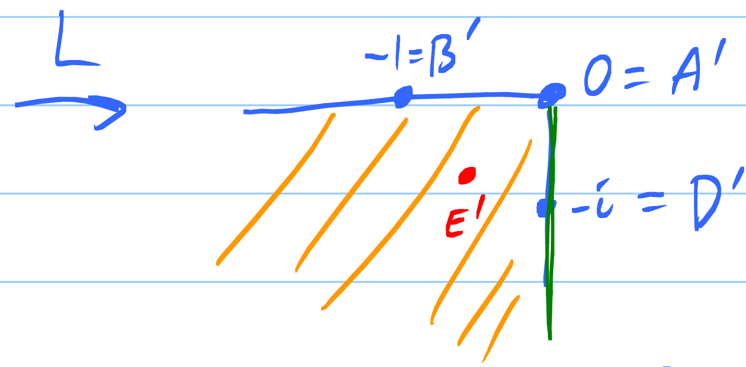
$L(-1) = 0$

$L(1) = \infty$

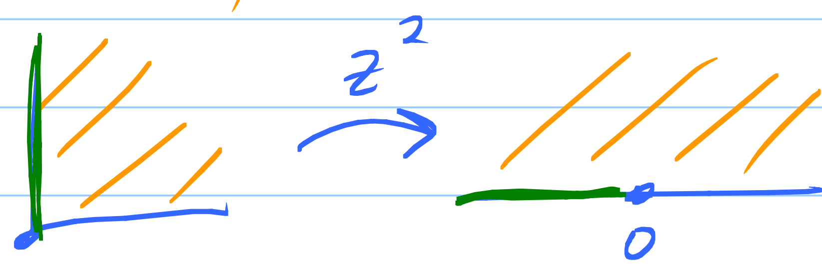
$L(i) = \frac{i+1}{i-1} = -i$

$L\left(\frac{i}{2}\right) = -\frac{3}{5} - \frac{4}{5}i$

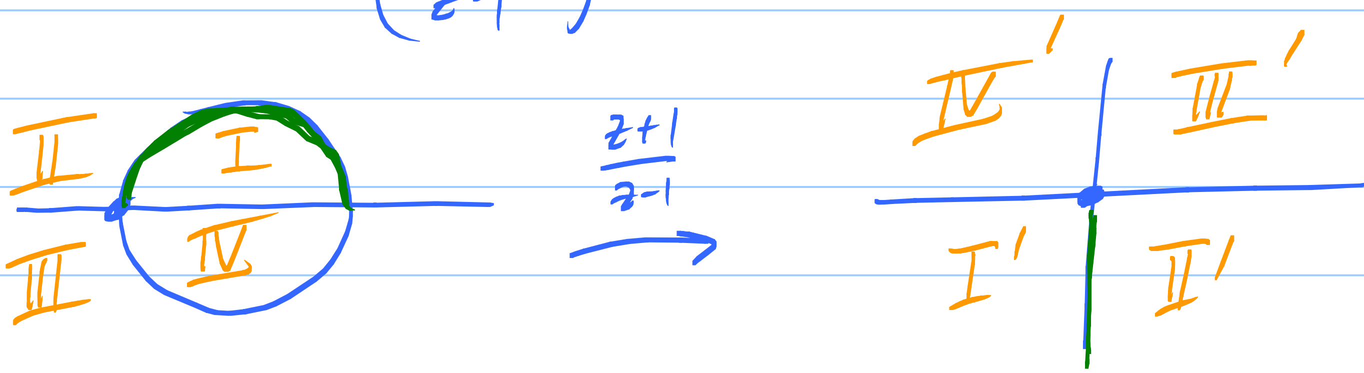




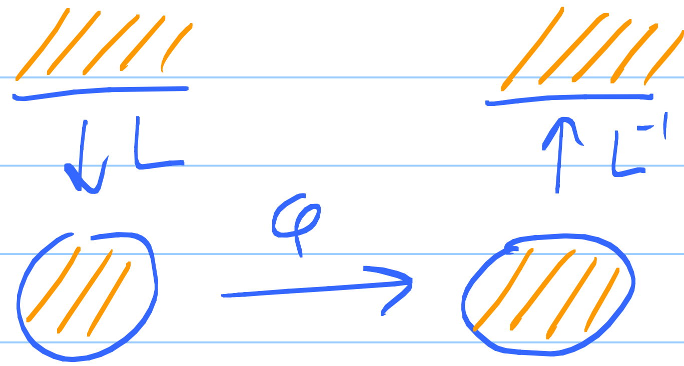
$e^{i\pi} z = -z$



Sol^n: $\left(\frac{z+1}{z-1}\right)^2$



Exercise: What is $\mathrm{Aut}(UHP)$?



$$\varphi(z) = \lambda \frac{z-a}{1-\bar{a}z}$$