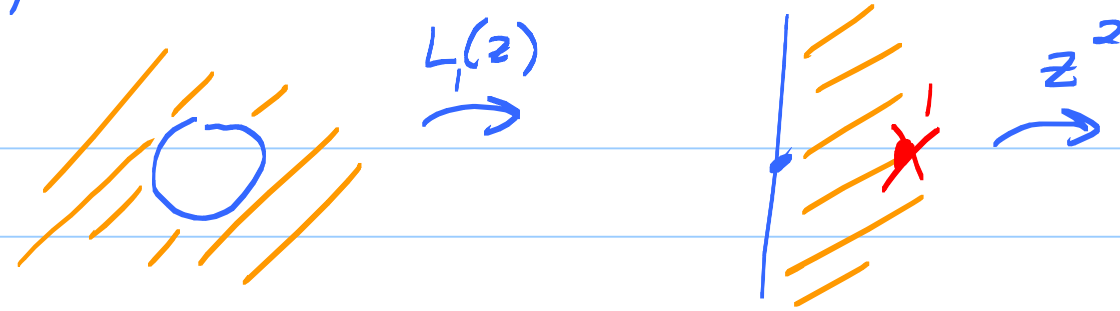
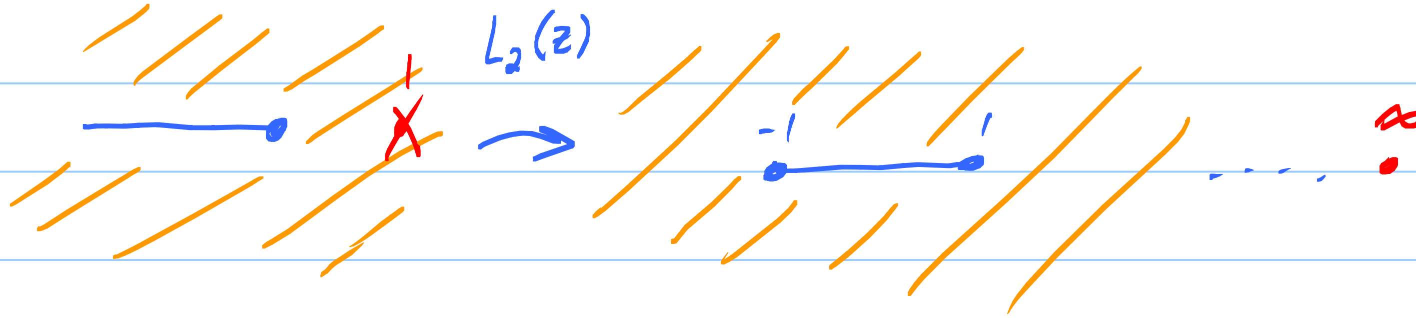
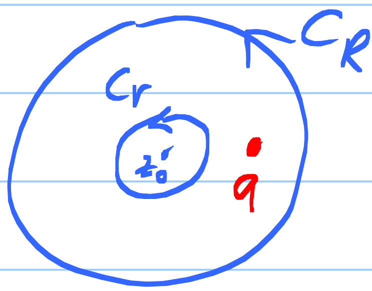
$J(z)$





$J(\infty) = \infty$.

__Cauchy Integral Formula for Annulus:__ Suppose f



analytic on $A(\rho_1, \rho_2; z_0)$ where $\rho_1 < r < R < \rho_2$.

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\, dz$$

__Pf:__ Apply Cauchy Theorem to $\dfrac{f(z)-f(a)}{z-a}$, which has a removable sing at $z=a$. So

$$\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)-f(a)}{z-a}\, dz = 0$$

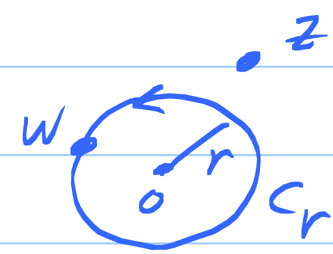
$$0 = \left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\,dz - f(a)\underbrace{\int_{C_R}\frac{1}{z-a}\,dz}_{2\pi i} + f(a)\underbrace{\int_{C_r}\frac{1}{z-a}\,dz}_{0}$$

Laurent Series: Take $z_0 = 0$,

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw}_{=\text{ power series}} - \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw}_{I(z)}$$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{(w-z_0)^{n+1}}\,dw$$



$$I(z) = -\frac{1}{2\pi i}\int_{C_r}\frac{-1}{z}\cdot\frac{f(w)}{\left(1-\frac{w}{z}\right)}\,dw$$

$$\left|\frac{w}{z}\right| < 1$$

$$\left[\text{Note: } \frac{1}{1-\tau} = \sum_{n=0}^{N}\tau^N + \underbrace{\frac{\tau^{N+1}}{1-\tau}}_{E_N(\tau)}\right]$$

$$I(z) = \frac{1}{2\pi i}\cdot\frac{1}{z}\int_{C_r} f(w)\left[\sum_{n=0}^{N}\left(\frac{w}{z}\right)^n + E_N\left(\frac{w}{z}\right)\right]dw$$

$$= \sum_{n=1}^{N+1} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} f(w)\, w^{n-1}\,dw\right)}_{a_{-n} = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w^{-n+1}}\,dw \quad \text{same!}} \frac{1}{z^n} + \mathcal{E}_N(z)$$

Finally, $\displaystyle |\mathcal{E}_N(z)| \le \frac{1}{2\pi}\,\frac{1}{|z|}\,\max_{C_r}|f| \cdot \frac{\left(\frac{r}{|z|}\right)^{N+1}}{1-\frac{r}{|z|}} \cdot 2\pi r$

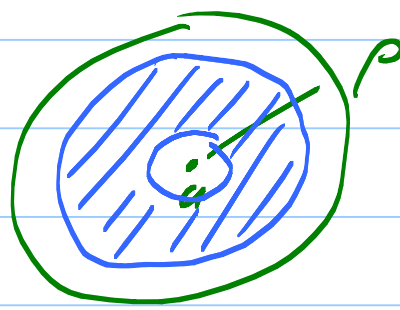
$\to 0$ uniformly if $|z| > \rho > r$.

__Theorem__: f analytic $D_\rho(a) - \{a\}$, then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\substack{\text{unif conv on}\\ |z-a| < R < \rho\\ \text{analytic on } D_\rho(a)}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} \frac{1}{(z-a)^n}}_{\substack{\text{unif conv on}\\ |z-a| > r > 0,\\ \text{analytic on } |z-a| > 0.}}$$

Note: $0 < r < R < \rho$ arbitrary.

Uniform convergence of both



__Def^n__: $\mathrm{Res}_a f = a_{-1}$

Theorem: 1) No negative terms in Laurent Expansion (LE) $\iff$ a is a removable singularity.

2) Only finitely many negative terms $\iff$ a is a pole.

3) Infinitely many negative terms $\iff$ a is essential.

Pf: (1) ($\Rightarrow$) $f(z)$ = power series. ✓

($\Leftarrow$) $a_{-n} = \frac{1}{2\pi i}\int_{C_r} \underbrace{f(w)(w-a)^{n-1}}_{\text{analytic in } C_r \text{ if } a \text{ removable}} dw \overset{\text{Cauchy}}{=} 0$

(2) Fact: Laurent Expansion is unique.

Why: $a_N = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{(z-a)^{N+1}} dz$

$C_\rho \leftarrow$ same for any ρ: $r \leq \rho \leq R$

$$= \frac{1}{2\pi i}\int_{C_\rho} \Big[\underbrace{\sum_{n=-\infty}^{\infty} b_n (z-a)^n}_{\text{unif conv on } C_\rho}\Big] \frac{1}{(z-a)^{N+1}} dz$$

$$= \sum_{n=-\infty}^{\infty} b_n \underbrace{\left(\frac{1}{2\pi i} \int_{C_\rho} (z-a)^{n-N-1} \, dz \right)}_{\text{all } = 0 \text{ except } n-N-1=-1, \text{ i.e., } n=N}$$

$$= b_N$$

(2) is done: $\underbrace{\sum_{n=1}^{N} \frac{a_{-n}}{(z-a)^n}}_{\text{Princ. Part}} + \sum_{n=0}^{\infty} a_n (z-a)^n$

N = order of pole ($a_{-N} \neq 0$).

(3) Left over case.

EX: $\sin \frac{1}{z} = 1 - \frac{(\frac{1}{z})^3}{3!} + \frac{(\frac{1}{z})^5}{5!} - \cdots$

Aha! a Laurent Exp. $z=0$ is an ess. sing.

Def^n: $\operatorname{Res}_a f = a_{-1} = \frac{1}{2\pi i} \int_{C_\rho} f(w) \, dw$

Aha! There will be a Residue Theorem for fcns that have ess. singularities.

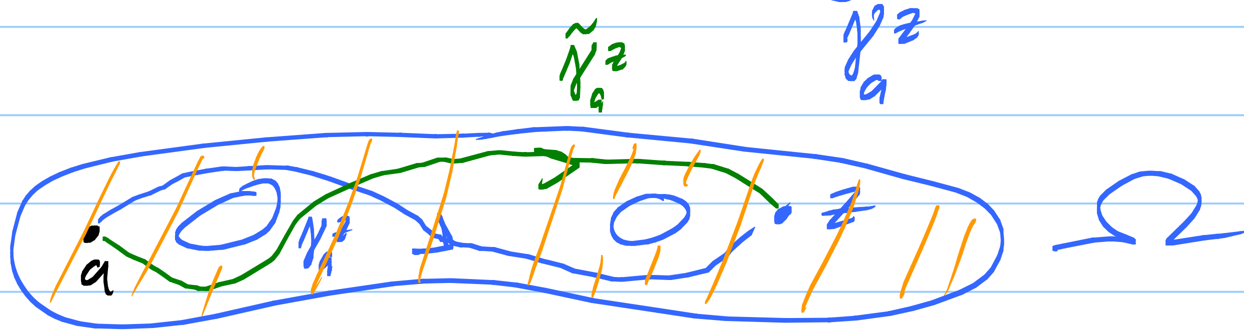
Meaning of residue: Suppose f is analytic on $D_R(a) - \{a\}$. f has an analytic antiderivative on $D_R(a) - \{a\} \iff \mathrm{Res}_a f = 0$.

Lemma: Suppose g analytic on a domain Ω. g has an analytic antiderivative $\iff$ $\int_\gamma g\,dz = 0$ for any closed curve γ in Ω.

Pf: $(\Rightarrow)$. $g = G'$. $\int_\gamma g\,dz = \int_\gamma G'\,dz =$

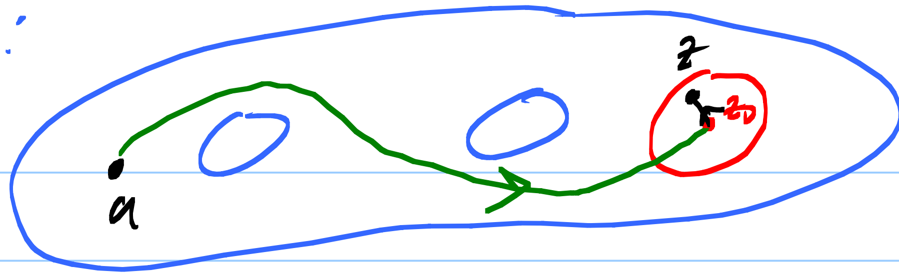
$= G(\text{END}) - G(\text{START}) = 0$ if γ closed.

$(\Leftarrow)$ "Define" $G(z) = \int_{\gamma_a^z} g(w)\,dw$



Well defined because $\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is closed in Ω. So $0 = \int_\Gamma g\,dw = \left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right) g\,dw = 0$.

Step 3:



$$G(z) = \underbrace{\int_{\gamma_a^{z_0}} g\,dw}_{\text{const.}} + \underbrace{\int_{L_{z_0}^{z}} g\,dw}_{\text{know derivative } = g(z)}$$

Done!

<u>Pf of Thm:</u> $(\Rightarrow)$ $f = F'$. $\text{Res}_a f = a_{-1} =$

$= \frac{1}{2\pi i}\int_{C_\rho} f\,dw = 0$

F'

$(\Leftarrow)$ Assume $\text{Res}_a f = 0$. Let γ be a closed curve in $D_R(a) - \{a\}$. Will show

$\int_\gamma f\,dz = 0$. Lemma $\Rightarrow f = F'$.

$$\int_\gamma f\,dz = \int_\gamma \sum_{-\infty}^{\infty} a_n (z-a)^n\,dz$$

tr(γ) compact

$$= \sum_{n=-\infty}^{\infty} a_n \underbrace{\left(\int_\gamma (z-a)^n \, dz \right)}_{\text{all } = 0, \text{ except maybe } n=-1. \text{ But } a_{-1}=0.}$$

$$= 0.$$