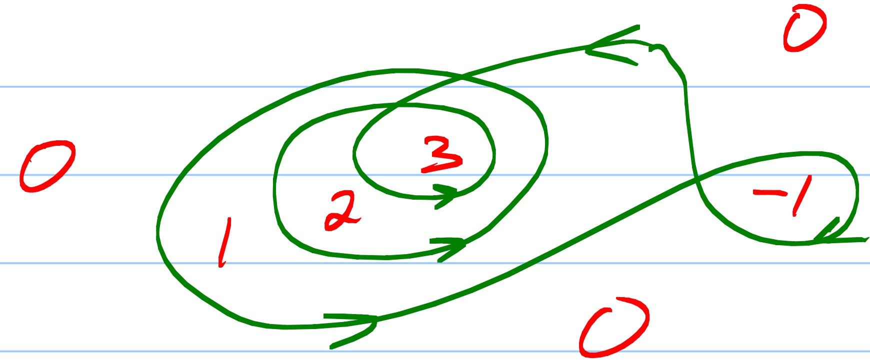
Index of a closed curve



$$\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \underbrace{\frac{1}{z-a}}_{\frac{f'}{f}}\,dz = \frac{1}{2\pi}\Delta_\gamma \arg(z-a) \in \mathbb{Z}$$

where $f(z) = z-a$

<u>Fact</u>: $\mathrm{Ind}_\gamma(z)$ is <u>constant</u> on connected components of $\mathbb{C} - \mathrm{tr}(\gamma)$.

<u>Why</u>: $$\frac{d}{dz}\left(\mathrm{Ind}_\gamma(z)\right) = \frac{1}{2\pi i}\int_\gamma \frac{1}{(w-z)^2}\,dw$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{d}{dw}\left[\frac{-1}{w-z}\right]dw = 0$$

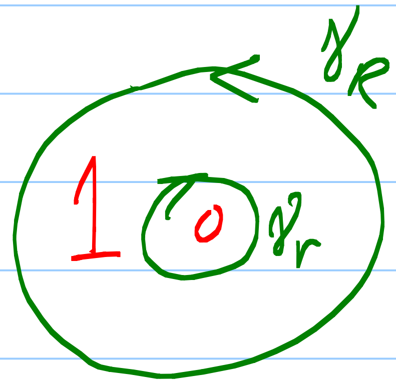
<u>Fact</u>: $\mathrm{Ind}_\gamma(z) \equiv 0$ on <u>unbounded</u> component of $\mathbb{C} - \mathrm{tr}(\gamma)$.

<u>Why</u>: $\mathrm{Ind}_\gamma(z) \to 0$ as $z \to \infty$.

<u>Def^{n}</u>: $\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$

is called a <u>cycle</u> if each γ_j is a closed curve.

<u>EX</u>: "Simple" cycle



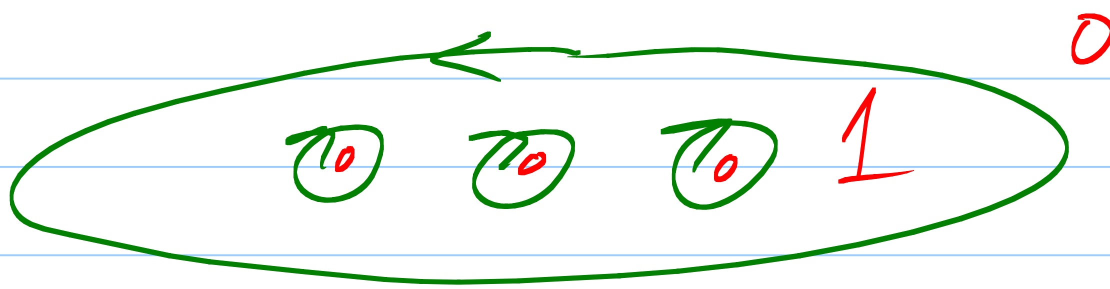
0

"Inside" = 1
"Outside" = 0

<u>Def^n</u>:

$$\int_\Gamma f\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f\,dz$$

$$\mathrm{Ind}_\Gamma(a) = \sum_{j=1}^{n} \mathrm{Ind}_{\gamma_j}(a)$$



<u>New version of Cauchy Theorem</u>: If f is analytic on a <u>simply connected</u> domain Ω and γ is a closed curve in Ω, then

1) $\int_\gamma f\,dz = 0$

2) $\mathrm{Ind}_\gamma(a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$

if $a \notin tr(\gamma)$.

<u>More General Residue Theorem</u>: If f has finitely many isolated singularities in (simply connected) Ω, then

$$\int_\gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Ind}_\gamma(a)\,\mathrm{Res}_a f$$

$A = \{$singular points in $\Omega\}$.

May use this for HWK 8, prob 2.

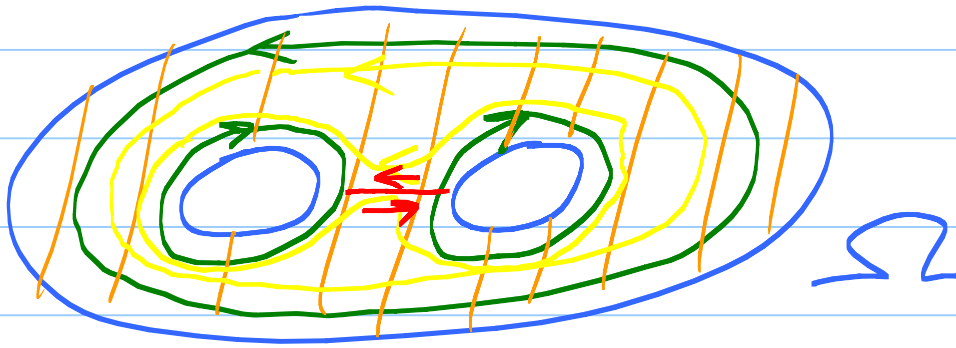
<u>General Cauchy Theorem</u>: Ω a domain, Γ is a cycle in Ω, f analytic on Ω, <u>and</u> $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$.

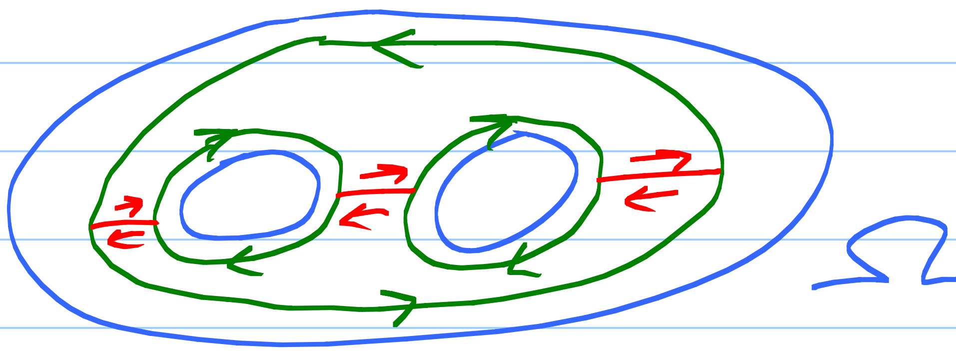
Then 1) $\displaystyle\int_\Gamma f\,dz = 0$

2) $\displaystyle \mathrm{Ind}_\Gamma(a) f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$

if $a \in \Omega - tr(\gamma)$.

Def^n: Γ is called "homologous to zero in Ω", write $\Gamma \sim 0$, if $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \mathrm{tr}(\gamma)$.





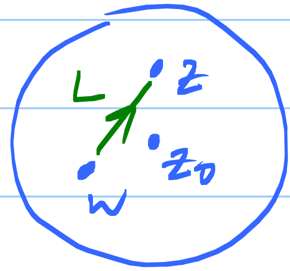
Pf of general Cauchy: (Ahlfors) Let

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

Step 1: g is continuous on $\Omega \times \Omega$.

Pf: Clear near (z_0, w_0) when $z_0 \neq w_0$.

Only need to check near (z_0, z_0).



$$g(z,w) - g(z_0, z_0) =$$

$$\frac{f(z)-f(w)}{z-w} - f'(z_0) =$$

$$\frac{1}{z-w} \int_L f'(\zeta)\,d\zeta - \underbrace{\frac{1}{z-w} \int_L f'(z_0)\,d\zeta}_{f'(z_0)(z-w)} =$$

$$\frac{1}{z-w} \int_L \left[f'(\zeta) - f'(z_0)\right] d\zeta$$

So $\displaystyle |g(z,w) - g(z_0,z_0)| \leq \frac{1}{|z-w|}\left(\max_L |f' - f'(z_0)|\right) \underbrace{|z-w|}_{\text{Length}}$

$\to 0$ as disc shrinks.

<u>Step 2</u>: Define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w)\,dw$$

<u>Plan</u>: Show that h extends to $\mathbb{C}$ as an <u>entire</u> fcn. $h \to 0$ at ∞. So $h \equiv 0$.

$$0 = h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw$$

$$= \frac{1}{2\pi i} f(z)\int_\Gamma \frac{1}{z-w}\,dw - \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\,dw$$

<u>Step 3</u>: h is analytic on Ω. (Morera's!)

1) Easy to see that h is <u>continuous</u> on Ω. [continuous g on $\Omega \times \Omega$, $tr(\Gamma)$ compact, uniform continuity on compact $\overline{D_\varepsilon(z)} \times tr(\Gamma)$, etc.]

2) $$\int_Q h\,dz = \int_Q \left(\int_\Gamma g(z,w)\,dw\right) dz$$

$$\underset{\text{Fubini}}{=} \int_\Gamma \left(\int_Q \underbrace{g(z,w)}_{} \,dz\right) dw$$

analytic in z!

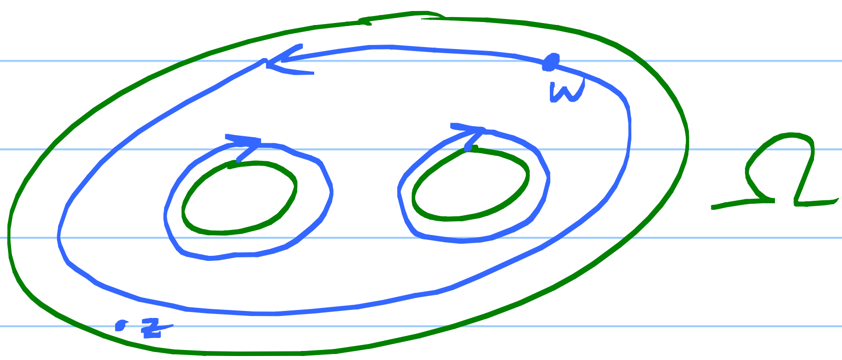
$$\begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

when w fixed.

$$= 0 \quad (\text{by Goursat}).$$

So h is analytic.

Step 4: Extend h to $\mathbb{C}$.



$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z) - f(w)}{z - w}\, dw$$

$$= \frac{1}{2\pi i} f(z) \underbrace{\int_\Gamma \frac{1}{z-w}\, dw}_{\substack{=0 \text{ for} \\ z \text{ outside} \\ \text{and for } z \text{ near boundary}}} - \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\, dw$$

Define $h(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\, dw$ for z "outside" Γ