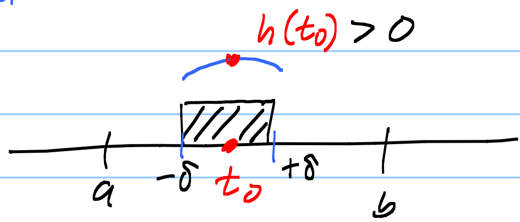
Lecture 11 The Maximum Principle

Calculus Lemma: If $h:[a,b]\to\mathbb{R}$ is continuous and $h(t)\geq 0$ on $[a,b]$, then

$$\int_a^b h(t)\,dt = 0 \implies h\equiv 0.$$

Pf:



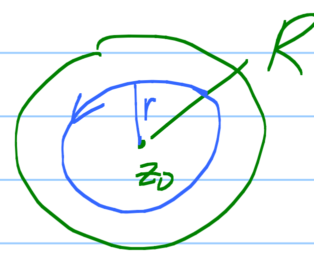
$\varepsilon = \frac{h(t_0)}{2}$ δ $\lightning$

Lemma: $h:[a,b]\to\mathbb{R}$ cont, $0\leq h(t)\leq M$, then

$$\int_a^b h(t)\,dt = M(b-a) \implies h\equiv M.$$

Pf: Let $\hbar = M-h$ in lemma above.

Averaging property for analytic fcns:



f analytic on $D_R(z_0)$:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt = \frac{1}{\underbrace{2\pi r}_{\text{Length } C_r}}\int_0^{2\pi} f(z_0+re^{it})\underbrace{r\,dt}_{ds}$$

Fact: Ave Property $\implies$ Max Princ.

Pf of Ave Prop:

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z_0}\,dw$$

C_r: $z(t) = z_0 + re^{it}$, $0\leq t\leq 2\pi$

$z'(t) = ire^{it}$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{it})}{(z_0+re^{it})-z_0}\,ire^{it}\,dt$$

$$= \text{average} \quad \checkmark$$

Maximum Modulus Theorem #1: Suppose f analytic on a domain Ω. If $|f|$ attains a local max at a pt $z_0 \in \Omega$, then f must be constant.

Note: $|f|$ local max at z_0 means $\exists\ \delta > 0$ such that $D_\delta(z_0) \subset \Omega$ and $|f(z)| \le \underbrace{|f(z_0)|}_{M}$ for $z \in D_\delta(z_0)$.

Pf: (Max Princ) Suppose z_0 is a local max for $|f|$. Get $D_\delta(z_0) \subset \Omega$ as above. Let $0 < r < \delta$. $|f(z_0)| = M$,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$$M = |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+re^{it})|}_{\le M}\,dt \le \frac{1}{2\pi}\cdot M \cdot 2\pi$$

Lemma #2 $\Rightarrow$ $|f(z_0+re^{it})| \equiv M$. r arbitrary!

So $|f(z)| \equiv M$ on $D_\delta(z_0)$. HWK prob $\Rightarrow$ $f \equiv$ const on $D_\delta(z_0)$. Uniqueness Thm $\Rightarrow$ $f \equiv$ const on Ω.

Max Princ #2: Suppose Ω is a bounded domain, and $f: \overline{\Omega} \to \mathbb{C}$ is continuous on the closure $\overline{\Omega}$, and f analytic on Ω. Then $|f|$ attains its max at a point on the boundary.

Pf: $\overline{\Omega}$ is closed and bounded, i.e., it is compact. f cont $\Rightarrow$ $|f|$ cont and cont fcn on compact assumes max M at $z_0 \in \overline{\Omega}$.

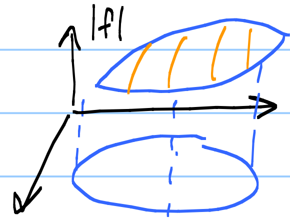
Case 1: z_0 in boundary. Done!

Case 2: $z_0 \in \Omega$. Max Princ #1 $\Rightarrow$ $f \equiv c$, const.

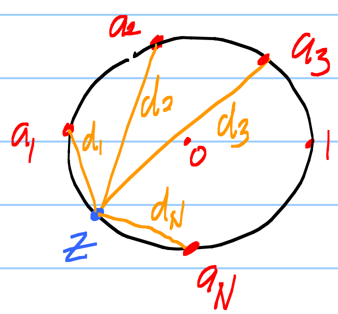
Claim: $f \equiv c$ on boundary too. Why: w_0 in boundry $\Rightarrow$

$\exists\ z_n \in \Omega$ with $z_n \to w_0 \notin \Omega$. Continuity of f $\Rightarrow$

$\underbrace{f(z_n)}_{c} \to f(w_0)$. So $|f(w_0)| = |c| = M$. ✓



EX:



Show $\exists\ z$ in unit circle where $\prod_{k=1}^{N} d_k = 1$.

Hmmm. $d_k = |z - a_k|$

$f(z) = \prod_{k=1}^{N} (z - a_k)$ ← not constant!

Aha! $f(0) = \prod_{k=1}^{N} (-a_k)$ $\quad |f(0)| = \prod_{k=1}^{N} \underbrace{|a_k|}_{1} = 1$

$\exists\ z_0$ in $C_1(0)$ where $|f(z_0)| = $ Max, which must be > 1

because f not const.

Continuous fcn $|f(e^{i\theta})|$ on $[0, 2\pi]$.

$\uparrow = 0$ when $e^{i\theta} = a_n$

and > 1 someplace.

Intermediate Value Thm yields $e^{i\theta}$ where it is exactly 1. ✓

Remark: z^n on $D_1(0)$ shows no Min. Princ..

But there is! Apply max princ to $1/f$.

Thm: Suppose f analytic on domain Ω and non-vanishing on Ω. Then Min. Princ. holds.

EX: Suppose f cont on $\overline{D_1(0)}$ and analytic inside, and non-vanishing on $D_1(0)$. If $|f(z)| \equiv 1$ when $|z|=1$, then $f \equiv \lambda$, a unimodular const.

Why: Max Princ $\Rightarrow$ $|f| \le 1$ on $\overline{D_1(0)}$.

Max Princ for $\frac{1}{f}$ $\Rightarrow$ $\frac{1}{|f|} \le 1$ on $\overline{D_1(0)}$.

So $|f| \equiv 1$. HWK $\Rightarrow$ f const λ on $D_1(0)$. $|\lambda| = 1$.

Theorem: Suppose u is a C^2-smooth $\mathbb{R}$-valued harmonic fcn on $D_R(z_0)$. Then u has a harm. conj v on $D_R(z_0)$ so that $f = u + iv$ is analytic. f infinitely complex diff'ble plus C-R eqns show that u and v are C^∞-smooth!