Lecture 12

Thm: u C^2-smooth harmonic fcn on $D_R(z_0)$, then $u = \mathrm{Re}\, f$ there, f analytic. So u is C^∞-smooth!

Pf: Suppose u C^2 and $\Delta u \equiv 0$. Want $f = u + iv$.

Need $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \longleftarrow \nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$

$\left[\text{Note: } \mathrm{Curl}\begin{pmatrix} -u_y \\ u_x \end{pmatrix} = \Delta u\, \hat{k} = 0, \text{ so potential } v\ \exists\right]$

Key: $v(z) - v(a) = \int_{\gamma_a^z} \nabla v \cdot d\vec{s} = \int_{\gamma_a^z} \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$

Idea: $v(z) \overset{\text{define}}{=} \int_{0 \to z} -u_y\,dx + u_x\,dy$

Assume $z_0 = 0$.

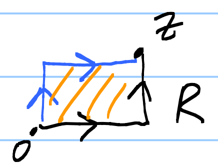
Step 1: Show $v_y = u_x$.

$$V = \underbrace{\int_{\to}}_{\text{const}} + \int_{x+i0}^{z = x+iy} \longleftarrow \quad \begin{aligned} x(t) &= x \\ y(t) &= t \qquad 0 \le t \le y \end{aligned}$$

So $v = (\text{Const}) + \int_0^y -u_y(x,t)\cdot \overset{\frac{dx}{dt}}{0}\cdot dt + \int_0^y u_x(x,t)\cdot \overset{\frac{dy}{dt}}{1}\cdot dt$

Fund. Thm Calc $\Rightarrow \frac{\partial v}{\partial y} = 0 + 0 + u_x(x,y)$ ✓

Step 2:



Green's Thm: $\int_{bR} \underset{-u_y}{P}\,dx + \underset{u_x}{Q}\,dy = \iint \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\Delta u} dx\,dy$

Aha! $v(z) = \int_{0 \to z}$

Step 3: Show $v_x = \frac{\partial}{\partial x} \int = -u_y$, F.T.C.

v is a harmonic conjugate for u. $f = u + iv$ analytic.

Conclude u C^∞-smooth!

Remark: Thm can proved for u on simply connected domains.

Define $v = \int_{\gamma_a^z} -u_y\,dx + u_x\,dy$.

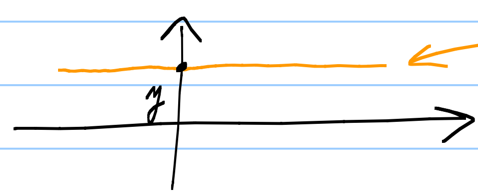
Warning: $\ln|z|$ is harmonic on $D_1(0) - \{0\}$, but it does not have a harm conj on set.

Lemma:

$$\frac{\partial u}{\partial x} \equiv 0 \iff u = \text{fcn of } y$$

$$\frac{\partial u}{\partial y} \equiv 0 \iff u = \text{fcn of } x$$

$\Longrightarrow$



$\frac{\partial u}{\partial x} \equiv 0 \Rightarrow u = \text{const}$ on horiz lines. Might depend on y.

Cor: $\frac{\partial u_1}{\partial x} \equiv \frac{\partial u_2}{\partial x}$, then $u_1 - u_2 = \text{fcn of } y$. etc.

Prob: Find a harmonic conj for $u = x^2 - y^2 + x + y$

Need v with

$$\frac{\partial v}{\partial x} = -u_y = 2y - 1 \qquad (A)$$

$$\frac{\partial v}{\partial y} = u_x = 2x + 1 \qquad (B)$$

Step 1: Use (A):

$$v = \int (2y-1)\,dx = 2xy - x + \underbrace{C(y)}_{\text{arb fcn of } y}$$

(treat y like a const)

Step 2: Use (B):

$$\frac{\partial}{\partial y}\left(2xy - x + C(y)\right) \overset{\text{want}}{=} 2x+1$$

$$2x - 0 + C'(y) = 2x+1$$

$$\boxed{C'(y) = 1}$$ ← all x's cancel!

So $C(y) = y + K$. Done!

$v = 2xy - x + (y+K)$ v uniq up to $+K$.

Fact: Harmonic fcns are locally the real part of analytic fcns.

EX: Zero sets of harmonic fcns?

$u(x,y) = y$ is harmonic, zero on $\mathbb{R}$. Zeroes not isolated, Uniqueness Thm will be different.

Thm: Suppose u is $\mathcal{C}^2$-smooth, $\Delta u \equiv 0$, on a domain Ω. If $u \equiv 0$ on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Important trick: u harmonic $\Rightarrow$

$F = u_x - i u_y$ is analytic.

[Think: $f = u + iv$. $f' = u_x + i v_x = u_x - i u_y$]

Pf Thm: analytic $F \equiv 0$ on $D_\varepsilon(z_0)$.

Uniqueness Thm $\Longrightarrow$ $F \equiv 0$ on Ω. $\Longrightarrow$ $\nabla u \equiv 0$ on Ω,

So $u \equiv c$, a const on Ω. $\left[\, u(z) - u(a) = \int_{\gamma_a^z} \nabla u \cdot d\vec{s} \,\right]$

Complex log fcns: $\log w = \{ z : e^z = w \}$, a set.

$e^z = e^{x+iy} = \underbrace{e^x e^{iy}}_{w} = w$ $\qquad e^x = |w| \qquad \boxed{x = \operatorname{Ln}|w|}$

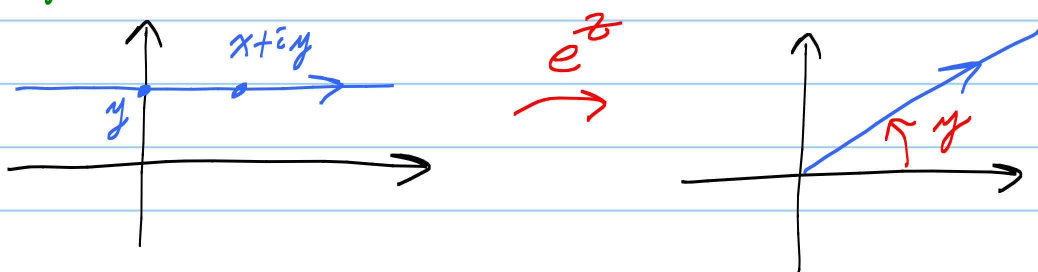
$y \in \arg w$

$\log w = \{ \operatorname{Ln}|w| + i\theta : \theta \in \arg w \} \leftarrow \infty$ many!

$\underbrace{\operatorname{Log}}\, w = \operatorname{Ln}|w| + i \operatorname{Arg} w \leftarrow$ Principal Arg

Principal branch of Log.

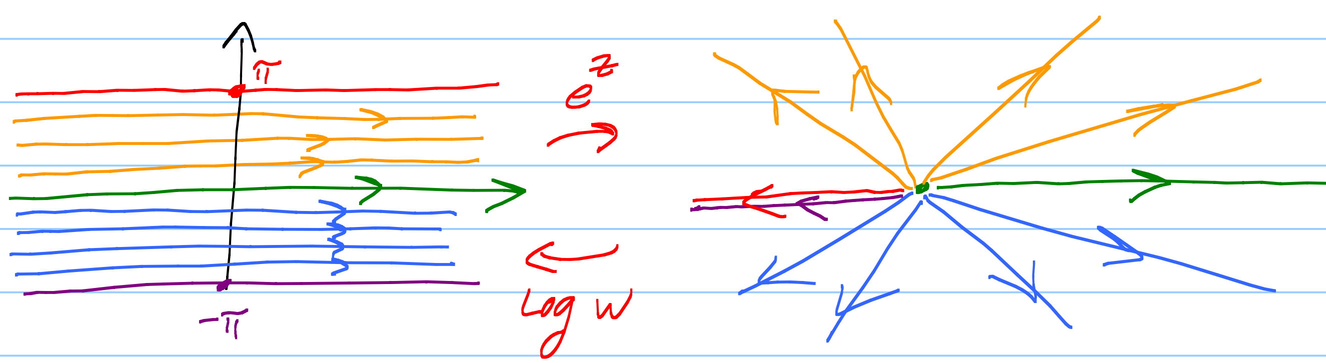
$\log w = \{ \operatorname{Ln}|w| + i(\operatorname{Arg} w + 2n\pi) : n \in \mathbb{Z} \}$

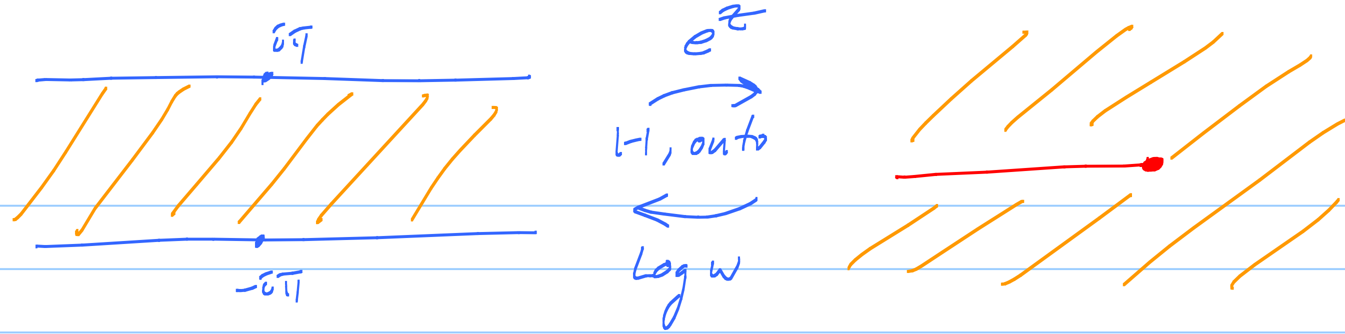


$e^{x+iy} = e^x e^{iy}$

$-\infty < x < \infty$

$0 < e^x < \infty$





$\{z = x + iy : -\pi < y < \pi\}$

$\mathbb{C} - (-\infty, 0]$