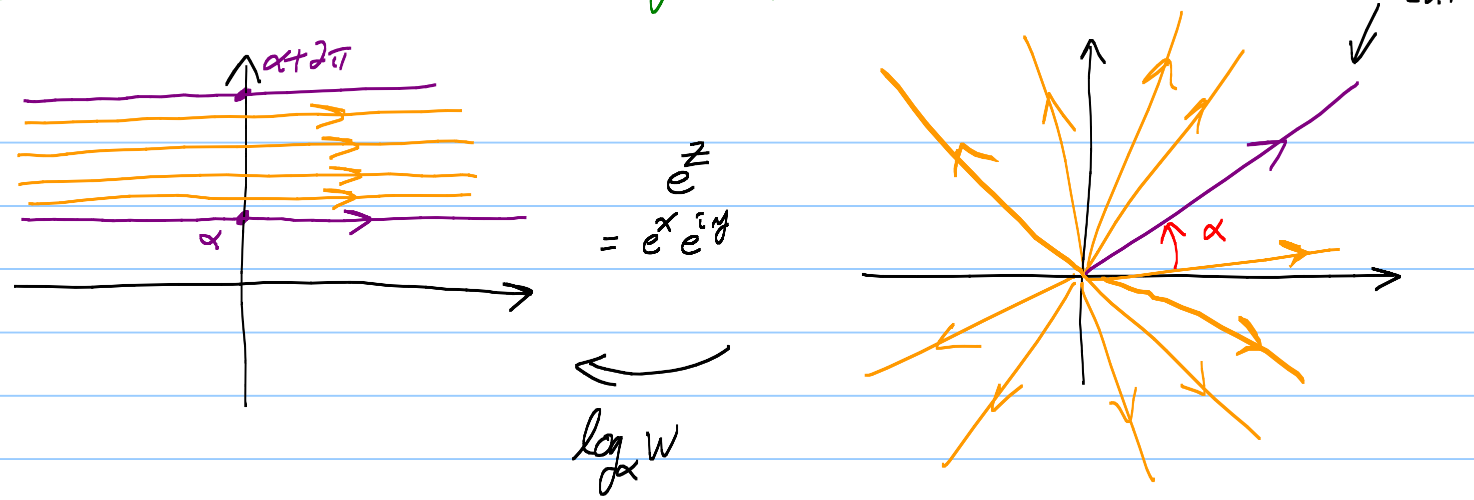
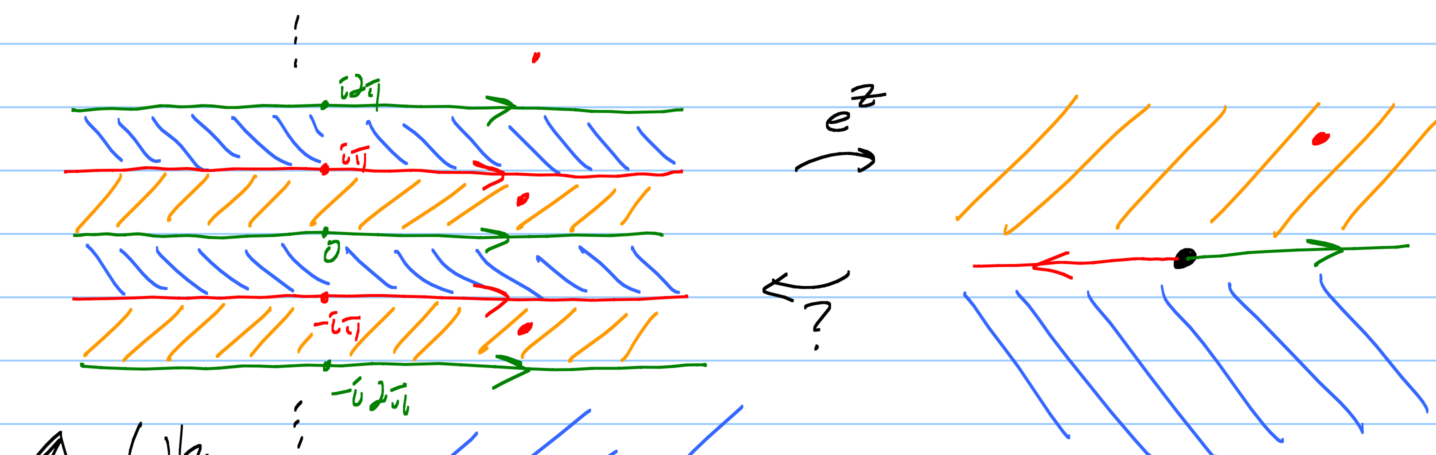
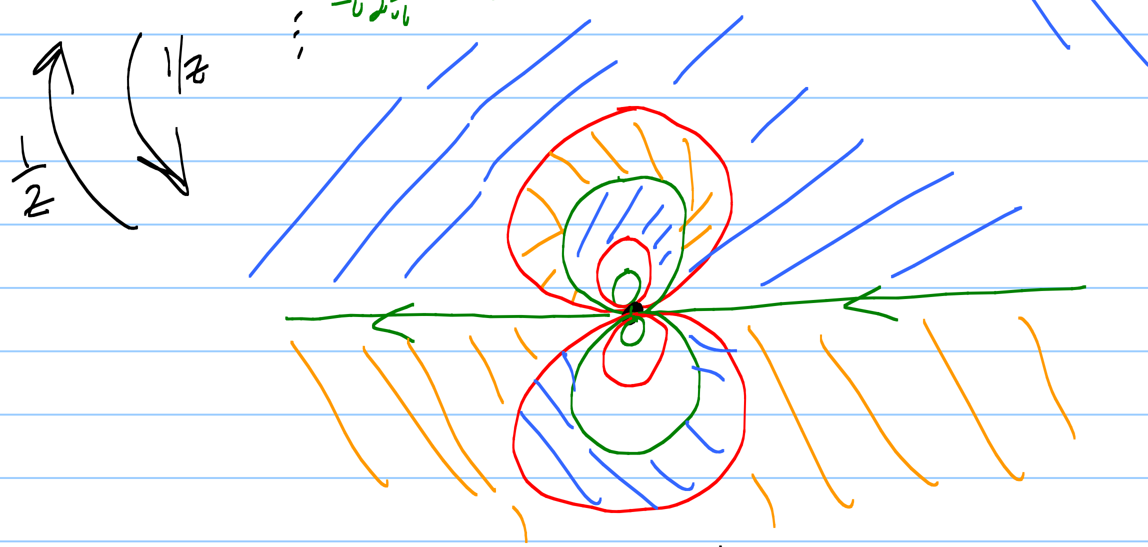
Lecture 13 Riemann Removable Singularity Theorem



$\log_\alpha w = \ln|w| + i\theta$ where $\theta \in \arg w$ with $\alpha < \theta < \alpha + 2\pi$

$=$ "branch of $\log w$"



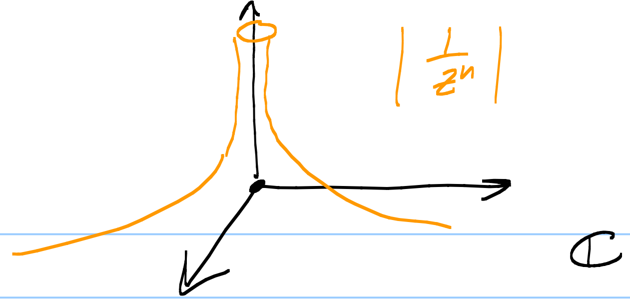


$e^{1/z}$ has a mind blowing singularity at $z=0$.

Now matter how small ε is $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ onto $\mathbb{C} - \{0\}$ ∞ to one! This is called an "essential singularity."

EX: 1) $e^{1/z}$ at $z=0$.

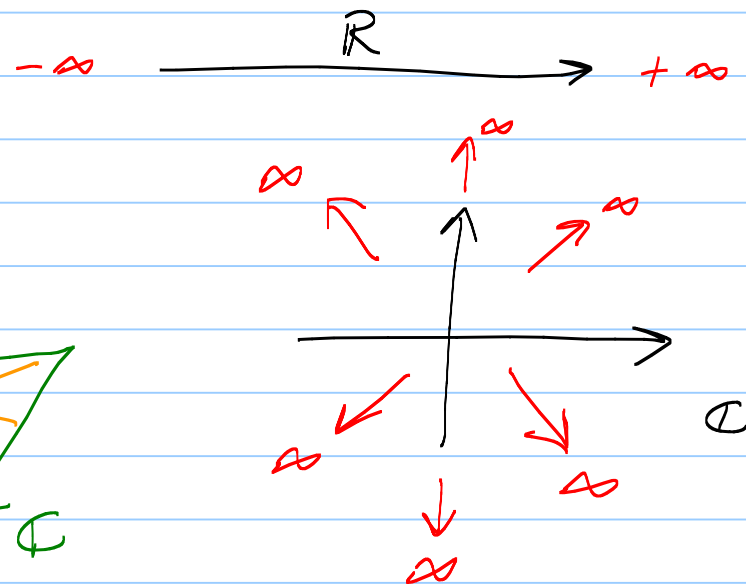
2) $\frac{1}{z^n}$ Modulus blows up as $z \to 0$. Has a "pole" at $z=0$



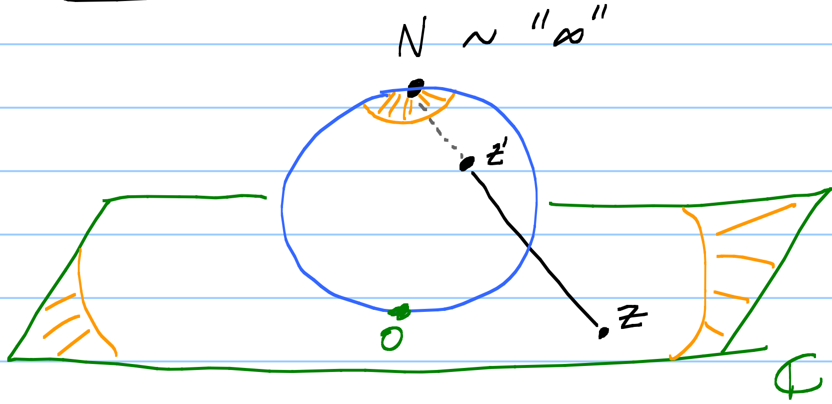
3) $$\frac{\sin z}{z} = \frac{\left(z - \frac{z^3}{3!} + \cdots\right)}{z} = \frac{z\left(1 - \frac{z^2}{3!} + \cdots\right)}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$ ← converges for all z. R. of C. $= \infty$.

$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$ is entire, $z=0$ is "removable"



The "point at infinity"



$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ = extended complex plane.

$$D_\varepsilon(\infty) = \{z : |z| > \tfrac{1}{\varepsilon}\} \cup \{\infty\}$$

Def^n: $\lim_{z \to a} f(z) = \infty$ means, given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $z \in D_\delta(a) - \{a\}$.

Remark: Setting $f(a) = \infty$ makes f continuous $D_\delta(a) \to \hat{\mathbb{C}}$ at a.

EX: $\lim_{z\to 0} \frac{1}{z^n} = \infty$.

Riemann Removable Sing Thm: Suppose f is analytic on $D_r(a) - \{a\}$ (f has an isolated singularity at a), and suppose f is bounded on $D_r(a) - \{a\}$ (say by M). Then f can be given a value at a that makes f analytic on $D_r(a)$.

Way false $\mathbb{R} \to \mathbb{R}$! $\quad \sin\frac{1}{t}$

Pf: Define
$$H(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$$

Claim: H analytic on $D_r(a)$.

$|f| < M$ shows H cont at a,

$$DQ = \frac{H(z) - H(a)}{z-a} = \frac{(z-a)^2 f(z) - 0}{z-a} = (z-a) f(z)$$
$$\to 0 \text{ as } z \to a.$$

and $H'(a) = 0$.

$$H(z) = \underset{\substack{\| \\ 0}}{a_0} + \underset{\substack{\| \\ 0}}{a_1}(z-a) + a_2(z-a)^2 + \cdots$$

$$= (z-a)^2 \Big[\overbrace{\underbrace{a_2 + a_3(z-a) + \cdots}_{\text{R. of. C} \geq r}}^{h(z) \text{ analytic on } D_r(a).} \Big]$$

Finally, $H(z) = (z-a)^2 h(z)$

$$f(z)(z-a)^2 = (z-a)^2 h(z)$$

Aha! $f(z) = h(z)$. $[f(a) = a_2]$

Alternate Pf: $G(z) = \begin{cases} (z-a)f(z) & z \neq a \\ 0 & z = a \end{cases}$

$a = p$ in Goursat's Lemma. G cont on $D_r(a)$, analytic on $D_r(a) - \{a\}$.

Morera's $\Rightarrow$ G analytic on $D_r(a)$. Finally, factor $z-a$ out of the power series.

Def^n: If a is an isolated sing of f, we say

f analytic on $D_\delta(a) - \{a\}$

1) a is removable if f can be given a value at a that makes f analytic on a disc about a.

2) a is a pole if $\lim_{z \to a} f(z) = \infty$.

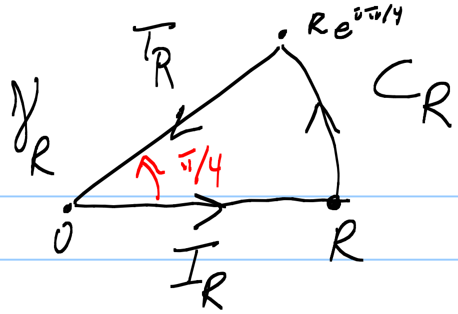
3) a is essential if, for every $\varepsilon > 0$ with $0 < \varepsilon < \delta$,

$$f(D_\varepsilon(a) - \{a\}) \text{ is dense in } \mathbb{C}.$$

Here, dense means: every pt in $\mathbb{C}$ is a limit point of $\{f(z) : z \in D_\varepsilon(a) - \{a\}\}$.

Thm: 1, 2, 3 are the only possibilities.

Hwk 4: Last problem.



$$\int_{\gamma_R} e^{-z^2}\, dz \qquad \text{Let } R \to \infty,$$

$I_R : z(t) = t \qquad \displaystyle\int_{I_R} e^{-t^2}\, dt$

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}}$$

Know.

$-T_R : \quad z(t) = te^{i\pi/4}, \quad 0 \le t \le R$

$\qquad z'(t) = e^{i\pi/4}$

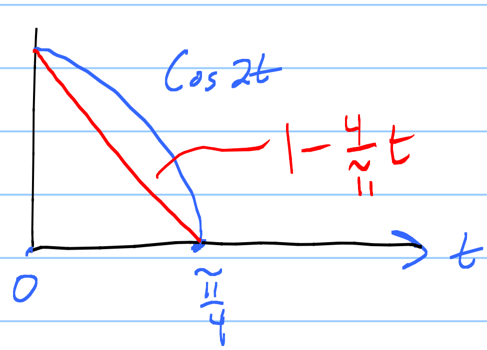
$\displaystyle\int_{T_R} = -\int_{-T_R} \quad \leftarrow$ want this!

<u>Claim</u> $\displaystyle\int_{C_R} \to 0$ as $R \to \infty$.

$C_R : z(t) = Re^{it} \qquad 0 \le t \le \frac{\pi}{4}$

$$\left| \int_{C_R} e^{-z^2}\, dz \right| = \left| \int_0^{\pi/4} e^{-(Re^{it})^2} (iRe^{it})\, dt \right|$$

$$\le \int_0^{\pi/4} e^{-R^2\cos 2t} \underbrace{\left| e^{-iR^2\sin 2t} \right|}_{1} \cdot R\, dt$$



red $\le \cos 2t$

$$e^{-R^2\cos 2t} \le e^{-R^2\left(1 - \frac{4}{\pi}t\right)}$$