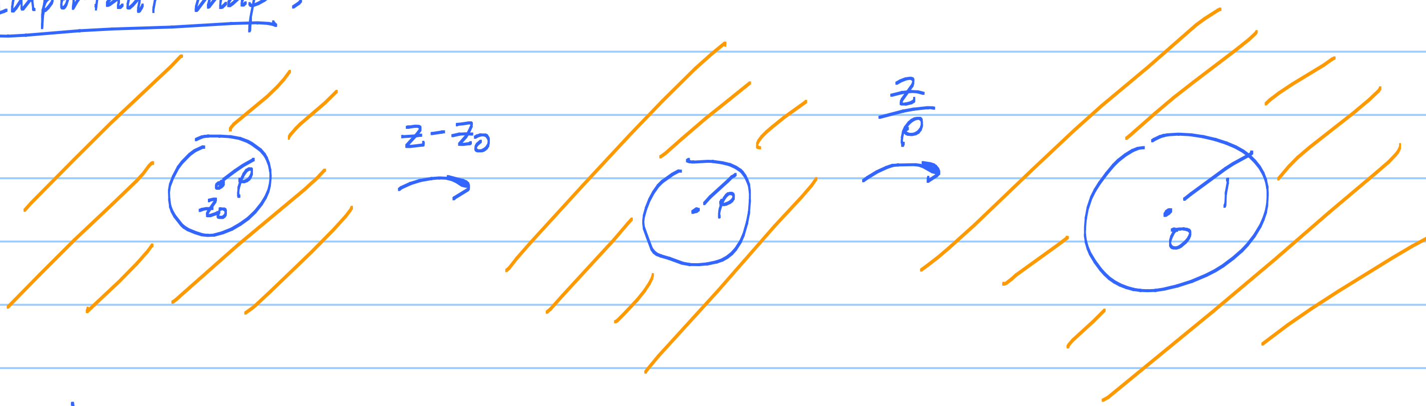
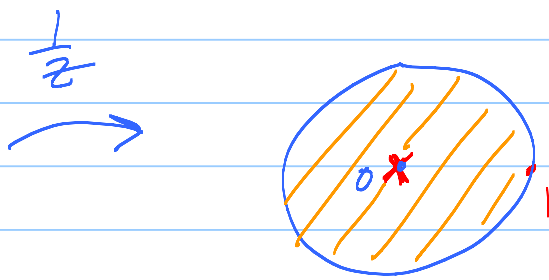
Lecture 14 Isolated singularities

Important map:





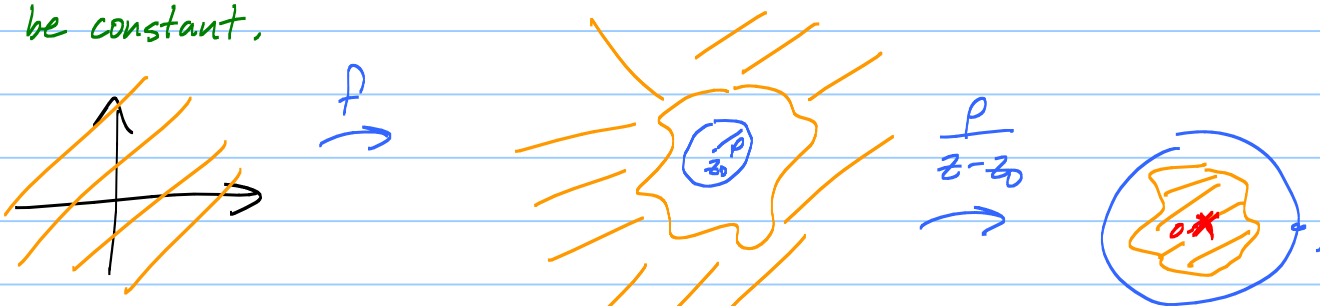
Composition: $\dfrac{\rho}{z - z_0}$

maps outside $\overline{D_\rho(z_0)}$

1-1 onto $D_1(0) - \{0\}$

Casorati-Weierstraß Thm: If an entire fcn misses an open set, then it must be constant.

Pf:



Aha! $\dfrac{\rho}{f(z) - z_0}$ is a bdd entire fcn. Liouville's $\Rightarrow$ const.

$\Rightarrow f \equiv$ const.

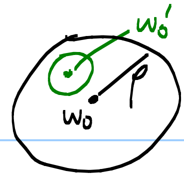
Thm: Only 3 kinds of isolated sing. 1) Removable, 2) Pole, 3) Essential.

Pf: Suppose f analytic on $D_r(a) - \{a\}$. And suppose a is not essential. Then $\exists\ \varepsilon$ with $0 < \varepsilon < r$ such

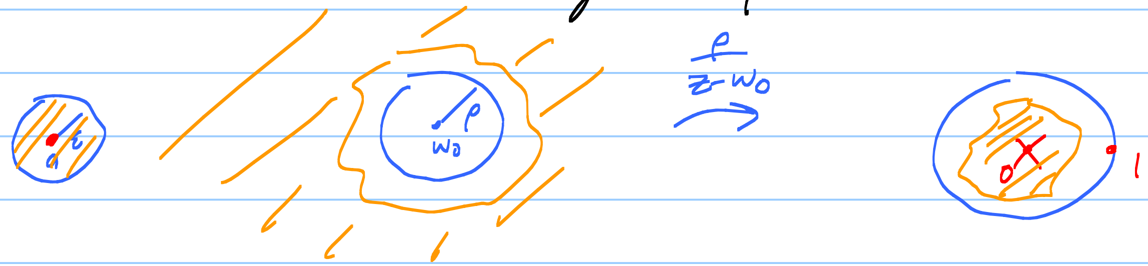
$S = f\left(D_\varepsilon(a) - \{a\}\right)$ is not dense in $\mathbb{C}$. Then $\exists\ w_0 \in \mathbb{C}$ that is not a limit pt of this set.

Case $w_0 \notin S'$: Then $\exists\ \rho > 0$ $D_\rho(w_0) \cap S' = \emptyset$.

Case $w_0 \in S'$:



Use w_0' in place of w_0 to get $D_\rho(w_0) \cap S' = \emptyset$.



Aha! $\dfrac{\rho}{f(z)-w_0}$ is bdd on $D_\varepsilon(a) - \{a\}$.

R.R.S.T. $\Longrightarrow$ a removable. Get

$$h(z) = \frac{\rho}{f(z)-w_0} \quad \text{with } h \text{ analytic on } D_\varepsilon(a).$$

Then $f(z) = w_0 + \dfrac{\rho}{h(z)}$ $\quad (*)$

Case $h(a) \neq 0$: $(*)$ shows that a is removable for f.

Case $h(a) = 0$: Then $(*)$ yields that $\lim_{z\to a} f(z) = \infty$.

f has a pole at a.

More details about poles:

Fact: $h(z)$ has a zero of order N at a

$\Longleftrightarrow$ $h(z) = (z-a)^N H(z)$ where $H(z)$ is analytic on a disc about a and $H(a) \neq 0$.

Poles: $\lim_{z\to a} f(z) = \infty$. Then $\exists\ r > 0$ such that

$|f(z)| > 1$ if $z \in D_r(a) - \{a\}$. So f has no zeroes here.

So $\frac{1}{f(z)}$ analytic on $D_r(a) - \{a\}$ and $\frac{1}{f(z)} \to 0$ as $z \to a$. RRST $\Longrightarrow$ $\frac{1}{f}$ has a removable sing at a.

$g(z) = \frac{1}{f(z)}$. $g(a) = 0$. Let N be the order of the zero. Then $g(z) = (z-a)^N G(z)$, $G(a) \neq 0$.

Now $f(z) = \frac{1}{g(z)} = \frac{1}{(z-a)^N G(z)} = \frac{1}{(z-a)^N} \cdot F(z)$

where $F(z) = \frac{1}{G(z)}$ is analytic on $D_r(a)$ and $F(a) \neq 0$.

Def^n: N is the order of the pole.

$$f(z) = \frac{1}{(z-a)^N} \underbrace{\left[A_0 + A_1(z-a) + \cdots \right]}_{F(z)}$$

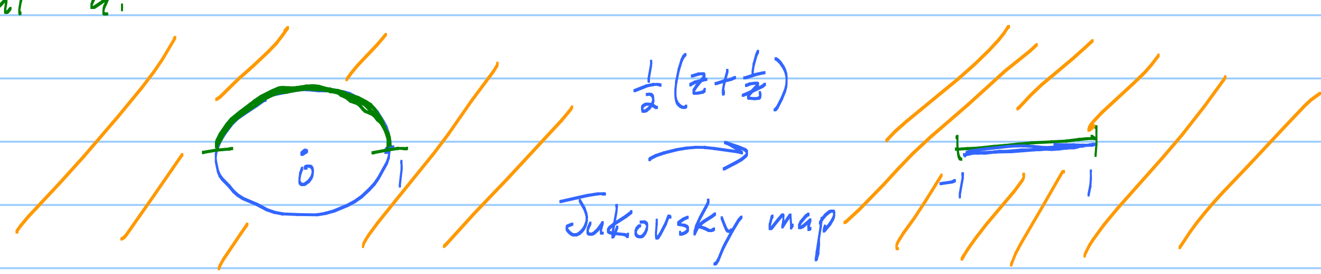
$$= \underbrace{\overset{\neq 0}{\frac{A_0}{(z-a)^N}} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{\text{Principal Part of } f \text{ at } a.} + \underbrace{A_N + A_{N+1}(z-a) + \cdots}_{\text{regular power series}}$$

Residue of f at a

R. of C. $\geq r$

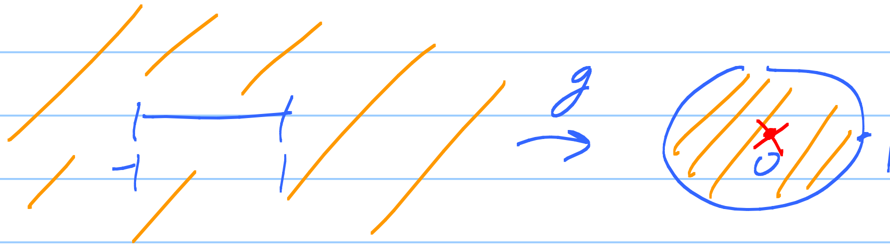
Def^n: $R(z) = \frac{A_0}{(z-a)^N} + \cdots + \frac{A_{N-1}}{z-a}$ is the unique fcn of this form such that $f(z) - R(z)$ has a removable sing at a.

Hmmm:



Inverse: $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ $\quad z = F(w)$ analytic.

Get analytic



Thm: An entire fcn that misses a line segment must be constant.

Picard's Little Theorem: Suppose f is entire. Then, either f is a polynomial, or f assumes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

Consequently, an entire fcn that misses two values in $\mathbb{C}$ must be const.

EX: e^z misses 0. $\sin z$ misses nothing.

Picard's Big Theorem: If f has an essential sing at a, then on any $D_\varepsilon(a) - \{a\}$ where f is analytic, f assumes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

EX: $e^{1/z}$ misses 0. $\sin \frac{1}{z}$ misses nothing.