Lecture 24 Linear Fractional Transformations (LFTs)

HWK 7 due Friday, March 24.

LFT: $L(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

[$ad-bc=0$: Then $L(z) \equiv$ const or not defined]

Lemma: LFTs are generated by

1) $z \mapsto rz$, $r \in \mathbb{R}^+$ (Stretch)
2) $z \mapsto e^{i\theta} z$ (Rotation)
3) $z \mapsto z+b$ (Translation)
4) $z \mapsto 1/z$ (Inversion)

Pf: $c=0$ case : $L(z) = \underset{\uparrow re^{i\theta}}{A}z + B$ 1, 2, 3

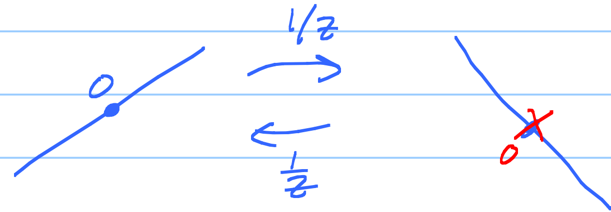
$c \neq 0$: $\dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{\overbrace{(b-\frac{ad}{c})}^{\leftarrow Re^{i\varphi}}}{\underset{\uparrow re^{i\theta}}{c}z+d}$

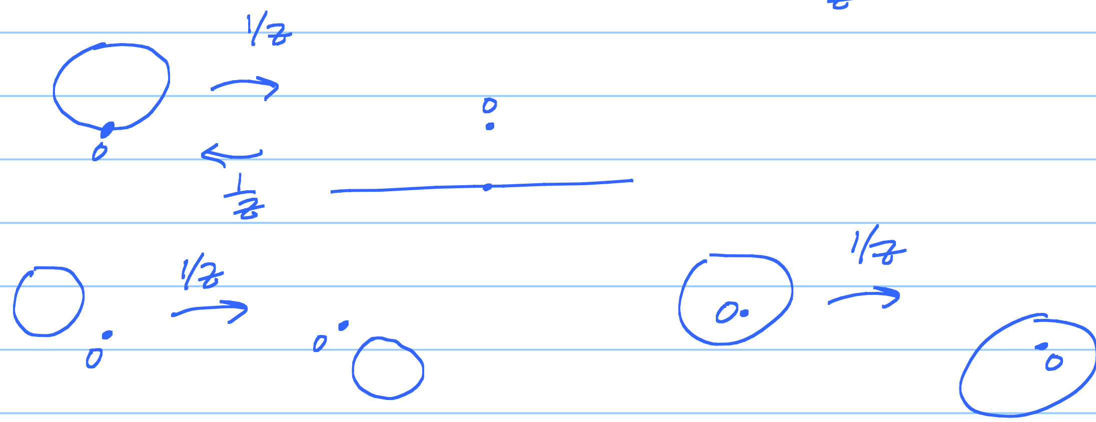
Need 7 basic maps.

Cor: LFTs take {lines, circles} $\circlearrowleft$.

Pf: 1, 2, 3 maps take {lines} $\circlearrowleft$ and {circles} $\circlearrowleft$.

4 can mix them up:





Fact: LFTs $\mathbb{C}\cup\{\infty\} \circlearrowleft$ 1-1 onto

LFTs = $\mathrm{Aut}(\hat{\mathbb{C}})$ ← {lines, circles} on $\hat{\mathbb{C}}$ are all just slices.

Note: $c \neq 0$: $L(z) = \dfrac{az+b}{cz+d}$ $\qquad \lim_{z\to\infty} L(z) = \dfrac{a}{c}$ $\qquad L(\infty) = \dfrac{a}{c}$

$\lim_{z\to -\frac{d}{c}} L(z) = \infty$ $\qquad L(-\frac{d}{c}) = \infty$

Analytic on a "polar cap": Use $\frac{1}{z}$ as a coord map.

$c=0$: $L(z) = Az + B$ $\qquad \lim_{z\to\infty} L(z) = \infty$ $\qquad L(\infty) = \infty$

Fact: $L \in$ LFT, then $L^{-1} \in$ LFT. ← They form a group.

Case $c=0$: easy.

Case $c \neq 0$: $w = \dfrac{az+b}{cz+d}$ ← solve for z $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Get $z = \dfrac{dw-b}{-cw+a}$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Fact: $\dfrac{az+b}{cz+d} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Group homomorphism $\qquad L_1 \circ L_2 \sim$ Matrix mult.

$L^{-1} \sim A^{-1}$

Lemma: A LFT that fixes 3 distinct pts must be the identity $L(z) = z$.

Pf: $L(z_j) = z_j$, $j = 1,2,3$

$\dfrac{az_j+b}{cz_j+d} = z_j$ $\qquad \boxed{c z_j^2 + (d-a) z_j - b = 0}$

Quadratic eqn with 3 roots! It must be $\equiv 0$.

$$\left.\begin{array}{l} a = d \\ c = 0,\ b = 0 \end{array}\right\} \quad L(z) = \frac{az}{d} = z \quad \checkmark$$

Cor: If two LFTs agree at 3 distinct pts, then they must be the same.

Note: 3 pts determine a circle!

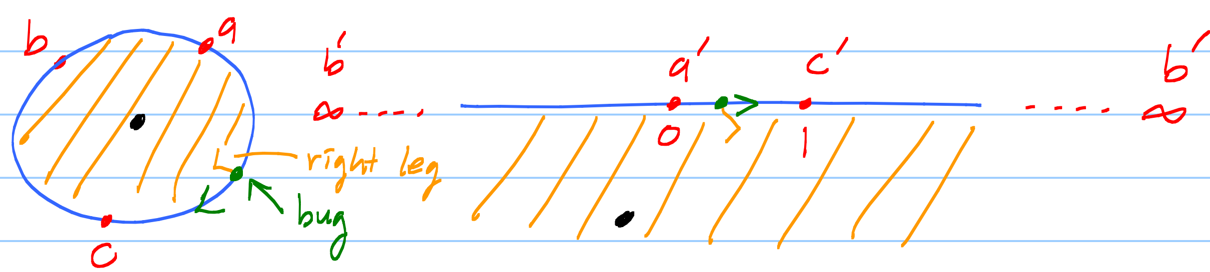
Only lines contain ∞.

Useful LFTs: EX: $\dfrac{z-a}{z-b}$ $\quad a \mapsto 0$, $\quad b \mapsto \infty$

Aha! Get $L(z)$: $\quad a \mapsto 0$, $\quad b \mapsto \infty$, $\quad c \mapsto 1$

$$L(z) = \frac{c-b}{c-a} \cdot \frac{z-a}{z-b}$$

EX:



Conformality (bug walk) $\Rightarrow$ inside $\to$ Lower Half Plane (LHP)

or just check a point inside.

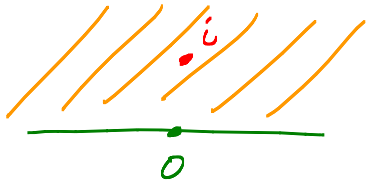
Prob: Find LFT: $\quad z_1 \mapsto w_1$, $\quad z_2 \mapsto w_2$, $\quad z_3 \mapsto w_3$

Sol^n: $L_1(z) = \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{z - z_1}{z - z_2}$ $\quad z_1 \mapsto 0$, $\quad z_2 \mapsto \infty$, $\quad z_3 \mapsto 1$

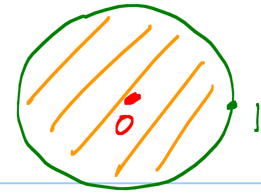
$L_2(w) =$ w's in place of z's

Sol^n: $L_2^{-1} \circ L_1$

EX:



$$L(z) = \frac{z-i}{z+i} \longrightarrow$$

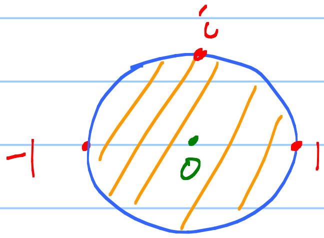


$$L : \text{UHP} \xrightarrow[\text{onto}]{1\text{-}1} D_1(0)$$

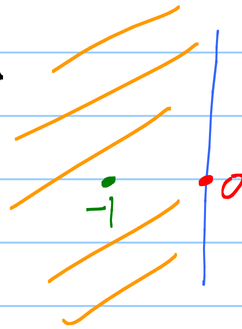
analytic

$$L^{-1}(w) = \frac{iz+i}{-z+1}$$

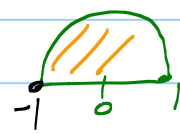
$$= -i\underbrace{\left(\frac{z+1}{z-1}\right)}_{\text{nice one too}}$$



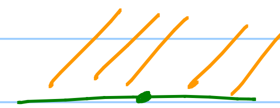
$$\xrightarrow{\frac{z+1}{z-1}}$$



Prob: Find a conformal map of

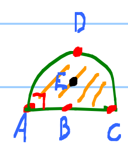


$\longrightarrow$

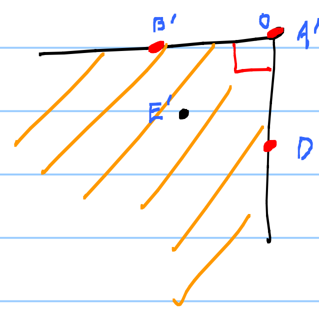


Idea: Blast a corner out to ∞.

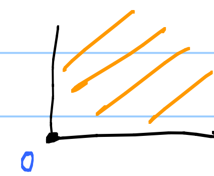
$$L(z) = \frac{z+1}{z-1}$$



$\xrightarrow{L(z)}$

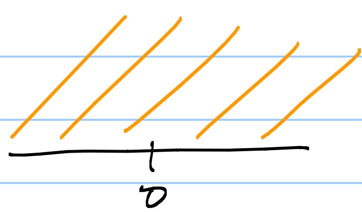


$\xrightarrow{-z}$

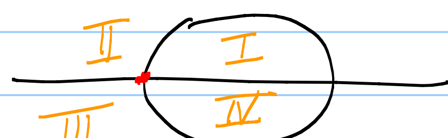


$L(0) = -1$
$L(-1) = 0$
$L(1) = \infty$
$L(i) = -i$
$L\left(\frac{i}{2}\right) = -\frac{3}{5} - \frac{4}{5}i$

$\xrightarrow{z^2}$



Riemann Sphere



$$\xrightarrow{\frac{z+1}{z-1}}$$