Lecture 32 D. Prob.

Solve D. Prob on $D_1(0)$. Given $\varphi(e^{i\theta})$ cont.
Cook up u continuous on $\overline{D_1(0)}$ with $u(e^{i\theta}) = \varphi(e^{i\theta})$ and u harmonic on $D_1(0)$.

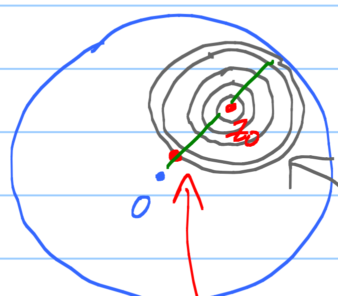
Plan: Last time, showed got polys $p_n(x,y)$ such that

$p_n(x,y) \to \varphi$ uniformly on $C_1(0)$.

Remark: Can use Stone-Weierstraß Thm instead of what I did.

Lemma: Averaging property $\Rightarrow$ weak global max princ on a disc: Given a continuous fcn u on $\overline{D_1(0)}$ that satisfies the Ave Prop in $D_1(0)$: If u attains its global max on $\overline{D_1(0)}$ at a point $z_0 \in D_1(0)$, then $u \equiv \text{Max}$ on $\overline{D_1(0)}$. Consequently, the max of such a fcn must be attained.

Pf:



$u(z_0) = M$, the max. Assume $M \geq 0$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + \rho e^{i\theta})}_{\leq M}\, d\theta$$

$u \equiv M$ on subdisc

Repeat at this z_0. Keep repeating until 0 is reached. $z_0 = 0$ shows $u \equiv M$ on $D_1(0)$.

Back to D. Prob: Solve D. Prob for P_n bndry data. Get harmonic polys $\underbrace{p_n(z) + q_n(\bar{z})}_{u_n}$ that do this.

Note: $u_n \to \varphi$ unif on $C_1(0)$. So u_n is uniformly Cauchy there. Max Princ $\Rightarrow$ unif Cauchy estimates extend to $\overline{D_1(0)}$. So $u_n \to$ uniformly on $\overline{D_1(0)}$ to a continuous fcn u. $\leftarrow$ Sol^n to D. Prob.

Note: $u = \varphi$ on $C_1(0)$.

<u>Claim:</u>

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\varphi(e^{i\theta})\,d\theta$$

$$= \mathrm{Re}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{\underbrace{e^{i\theta}}_{w}-z}\,\varphi(e^{i\theta})\,\frac{\overbrace{ie^{i\theta}\,d\theta}^{dw}}{i\underbrace{e^{i\theta}}_{w}}$$

$$= \mathrm{Re}\left[\underbrace{\int_{C_1(0)}\frac{\sigma(w)}{w-z}\,dw + z\int_{C_1(0)}\frac{\lambda(w)}{w-z}\,dw}_{\text{analytic in } z.}\right]$$

Consequently, $u = \mathrm{Re}\,F(z)$ on $D_1(0)$, and so it is <u>harmonic</u>.

<u>Pf of claim:</u> Bergman–Szegő kernel idea:

$z^n, \bar{z}^n, 1$ are $\perp$ on $C_1(0)$:

$$\langle z^n, \bar{z}^m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\,\overline{e^{-im\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n+m)\theta}\,d\theta$$
$$= 0$$

Bergman kernel idea: φ_m orthogonormal

$$K(z,w) = \sum_{m=1}^{\infty}\varphi_m(z)\,\overline{\varphi_m(w)} \qquad w = e^{i\theta},\ z\in D_1(0)$$

$$\frac{1}{2\pi}\int_0^{2\pi} K(z,w)\,\varphi_n(w)\,d\theta = \sum_{m=1}^{\infty} \varphi_m(z)\,\underbrace{\frac{1}{2\pi}\int_0^{2\pi} \overline{\varphi_m(w)}\,\varphi_n(w)\,d\theta}_{\substack{=0 \text{ if } n\neq m, \\ =1 \text{ if } n=m.}}$$

$$= \varphi_n(z)$$

$$K_N(z,w) = \sum_{n=1}^{N} \bar{z}^n w^n + \sum_{n=0}^{N} z^n \bar{w}^n$$

$$= (\bar{z}w)\sum_{n=0}^{N-1} \bar{z}^n w^n + \frac{1-(z\bar{w})^{N+1}}{1-z\bar{w}}$$

$$= \bar{z}w\,\frac{1-(\bar{z}w)^N}{1-\bar{z}w} + \frac{1-(z\bar{w})^{N+1}}{1-z\bar{w}}$$

$$\longrightarrow \underbrace{\frac{\bar{z}w}{1-\bar{z}w} + \frac{1}{1-z\bar{w}}}_{K(z,w) \;=\; \mathrm{Re}\left[\frac{1+\bar{w}z}{1-\bar{w}z}\right]} \qquad \text{as } N\to\infty \text{ since } z\in D_1(0).$$

<u>Note</u>: $\frac{1}{2\pi}\int_0^{2\pi} K_N(z,w)\,[\text{Harmonic Poly}]\,d\theta = \text{Harm Poly}$ for $N >$ degree.

Since $K_N \to K$ unif on $C_1(0)$ for fixed z, see that

$$\frac{1}{2\pi}\int K(z,w)\,[\text{Harm Poly}]\,d\theta = [\text{Harm Poly}].$$

Take limits $u_n \to u$. See

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z,e^{i\theta})\,\varphi(e^{i\theta})\,d\theta. \qquad \text{Done!}$$

Fact: Solutions to D. Prob with poly bndry data are polynomial.

Also true for ellipses.

Khavinson-Shapiro Conjecture: Only Ω's that have this property.

What if bndry data is rational on $D_1(0)$?

$$R(x,y) = R\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = Q(z,\bar{z}) \leftarrow \bar{z} = \frac{1}{z} \text{ on } C_1(0)$$

$$= Q\left(z, \frac{1}{z}\right) \leftarrow \text{on } C_1(0). \text{ Rational!}$$

$$= \text{Partial Fractions decom.} \leftarrow \text{no poles on } C_1(0)$$

$$= \underbrace{\sum_{a_k \in D_1(0)} \frac{A_{nk}}{(z-a_k)^n}}_{} + \underbrace{\sum_{|b_\ell|>1} \frac{B_{m\ell}}{(z-b_\ell)^m} + \underset{\uparrow \text{ poly}}{P(z)}}_{\text{analytic in } z \text{ on disc bigger than } D_1(0)}$$

$$\sum \frac{A_{nk}}{\left(\frac{1}{z} - a_k\right)^n}$$

$$\underbrace{\sum \frac{A_{nk}\,\bar{z}^n}{(1-\bar{z}a_k)^n}}_{}$$

$$= \overline{r(z)} \leftarrow \text{rational with no poles in } D_1(0)$$

Conclude: Sol^n to D. Prob is rational on $D_1(0)$.

B., Ebenfeld, Khavison, Shapiro: Discs only domains with this property!