Lecture 33 The Poisson kernel

$$K_N(z,w) = \sum_{n=1}^{N} \bar{z}^n w^n + \sum_{n=0}^{N} z^n \bar{w}^n$$

$w = e^{i\theta}$

$z \in D_1(0) : |z| < 1$

$$= \bar{z}w \cdot \frac{1-(\bar{z}w)^N}{1-\bar{z}w} + \frac{1-(z\bar{w})^{N+1}}{1-z\bar{w}}$$

$$\downarrow$$

$$K(z,w) = \frac{\bar{z}w}{1-\bar{z}w} + \frac{1}{1-z\bar{w}}$$

$$= \frac{\frac{1}{2}(1+\bar{z}w)+\frac{1}{2}(-1+\bar{z}w)}{1-\bar{z}w} + \frac{\frac{1}{2}(1+z\bar{w})+\frac{1}{2}(1-z\bar{w})}{1-z\bar{w}}$$

$$= \frac{1}{2}\left[\frac{1+\bar{z}w}{1-\bar{z}w} + \frac{1+z\bar{w}}{1-z\bar{w}}\right] - \frac{1}{2} + \frac{1}{2}$$

$$= \mathrm{Re}\left[\frac{1+z\bar{w}}{1-z\bar{w}}\right] \qquad w = e^{i\theta}, \quad \bar{w} = e^{-i\theta}$$

$$= \mathrm{Re}\left[\frac{w+z}{w-z}\right]$$

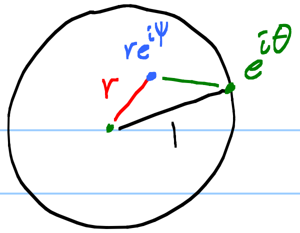
Sol^n to D. Prob is given by

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] \varphi(e^{i\theta})\, d\theta$$

Aha! Harmonic conjugate operator :

$$u \longmapsto v = \mathrm{Im}\, \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \varphi(e^{i\theta})\, d\theta$$

Poisson kernel : $\quad P(z,\theta) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$



$$= \frac{1}{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \qquad z = re^{i\psi}$$

$$= \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}$$

Nice fact: A domain bounded by a quadratic:

$\Omega = \{ r(x,y) < 0 \}$ $\quad r(x,y) = Ax^2+By^2+Cxy+Dx+Ey+F$

(Ω = circle or ellipse) satisfies (Poly data) $\to$ (Poly solutions to D. Prob.). [Khavinson-Shapiro Conj : Only ones]

Pf: $\mathcal{P}_N$ = polys in x and y of deg N. Finite dim^{e} vector space.

Fisher map: $p \longmapsto \Delta(rp)$ $\quad \circlearrowleft \mathcal{P}_N$ $\quad$ Linear

$\tilde{\Delta}$ $\qquad$ Δ: order 2 $\qquad$ r: deg 2

Claim: Fisher map is 1-1 :

Suppose $\tilde{\Delta}(p) = 0$

$\Delta(rp) = 0 \leftarrow$ So rp is harmonic.

But $r=0$ on bndry.

Max Princ $\Rightarrow$ $rp \equiv 0$.

But $r<0$ inside Ω. So $p \equiv 0$. ✓

Consequently $\tilde{\Delta} : \mathcal{P}_N \xrightarrow{1-1} \mathcal{P}_N$ is onto. [Linear algebra!]

How to solve D. Prob: Given poly q.

$\tilde{\Delta}$ onto. So $\exists\ p$ with $\Delta(rp) = \underbrace{\Delta q}_{\text{poly}}$

$q - rp$ solves the D. Prob.

1) $q - rp = q$ on bndry ✓ $\quad r=0$ on bndry

2) $\Delta(\underbrace{q-rp}_{\text{harmonic}}) = \Delta q - \Delta(rp) \equiv 0$ ✓

<u>Proof that Ave Prop $\Rightarrow$ harmonic:</u> Suppose u satisfies ave prop on $D_R(0)$ with $R>1$, u continuous. Define

$$\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta.$$

Know $\tilde{u}$ is continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$, <u>and</u> $\tilde{u} = u$ on $C_1(0)$.

Harmonic fcns satisfy Ave Prop. So $u - \tilde{u}$ satisfies ave prop. But Ave Prop $\Rightarrow$ Max, Min Princ and $u - \tilde{u}$ is <u>zero</u> on the bndry. So $u - \tilde{u} \equiv 0$.

Aha! $u = \tilde{u} \leftarrow$ harmonic!

<u>The classic way of solving the D. Prob</u> Schwarz

Notation abuse: $\varphi(\theta)$, $\varphi(e^{i\theta})$, $\varphi(x,y)$ all stand for same thing: A continuous fcn on $C_1(0)$. $\quad\varphi(z,\bar{z})$ too!

Problem: Look for a "Cauchy Integral Formula" for harmonic fcns.

f analytic: $\qquad f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta$

$u = \operatorname{Re} f \qquad u(0) = \operatorname{Re} \int = \int \operatorname{Re} f$

So $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$

Write $\varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z}$. $\varphi_{-a}(0) = a$

Fact: $\underbrace{\text{Harmonic fcn}}_{=\operatorname{Re} f}$ composed with an $\underbrace{\text{analytic fcn}}_{g}$ is harmonic. $\operatorname{Re}[f\circ g]$

locally.

$u \circ \varphi_{-a}$ is harmonic. Ave Prop:

$$u(a) = (u \circ \varphi_{-a})(0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\underbrace{\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}}_{e^{i\psi}}\right) d\theta$$

Hairy calculus exercise: $d\theta = \frac{1-|a|^2}{|e^{i\psi}-a|^2}\,d\psi$

Get Poisson Formula. (p. 66: Problem 11 in Stein)

We have: u harmonic on $D_R(0)$, $R>1$, then

$$u(z) = \int_0^{2\pi} P(z,\theta)\,u(e^{i\theta})\,d\theta.$$

Standard trick: u continuous on $\overline{D_1(0)}$, harmonic inside. Then Poisson Formula holds.

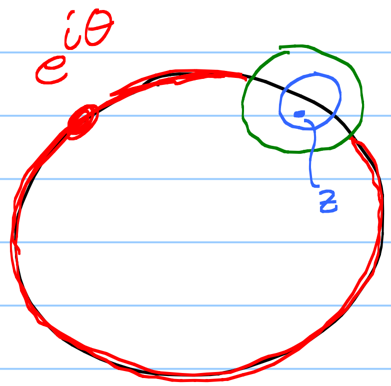
Let $\tilde{u}(z) = u(\rho z)$ where $0<\rho<1$. $\tilde{u}$ harm on $D_R(0)$. $\tilde{u}$ = Poisson Formula. Now let $\rho \nearrow 1$. Use unif continuity, etc.

Schwarz Thm: Poisson Formula solves D. Prob.

Key properties of $P(z,\theta)$:

1) $P(z,\theta) > 0$ in $z \in D_1(0)$

2) $\displaystyle\int_0^{2\pi} P(z,\theta)\, d\theta = 1$ for $z \in D_1(0)$.

3) 

$P(z,\theta)$ is really small on red part. $\to 0$ unif as $z \to$ bndry.

4) $P(z,\theta)$ is harmonic z.