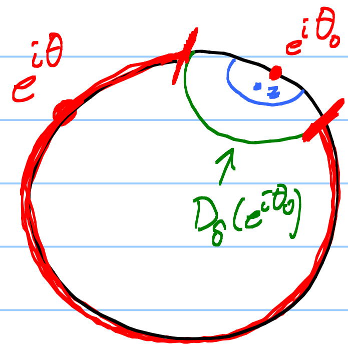
Lecture 34 Schwarz Theorem

$$P(z,\theta) = \frac{1}{2\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] = \frac{1}{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

Properties: 1) $P(z,\theta) > 0$, $z \in D_1(0)$ ← ✓ Formula

2) $\int_0^{2\pi} P(z,\theta)\, d\theta = 1$, $z \in D_1(0)$ ← ✓ Poisson formula for $u(z) \equiv 1$.

3)



4) $P(z,\theta)$ harmonic in z ✓ formula

$z \in D_{\delta/2}(e^{i\theta_0})$

$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

If $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$ and $\theta \in I_\delta$, then

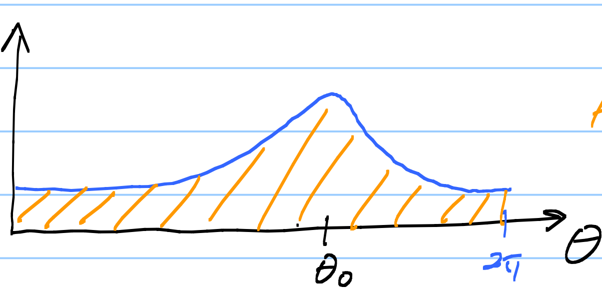
$$P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$$

Pf of 3: $P(z,\theta) = \dfrac{1-|z|^2}{2\pi|e^{i\theta}-z|^2} \le \dfrac{(1-|z|)\overbrace{(1+|z|)}^{\le 2}}{2\pi\,(\delta/2)^2}$ ✓

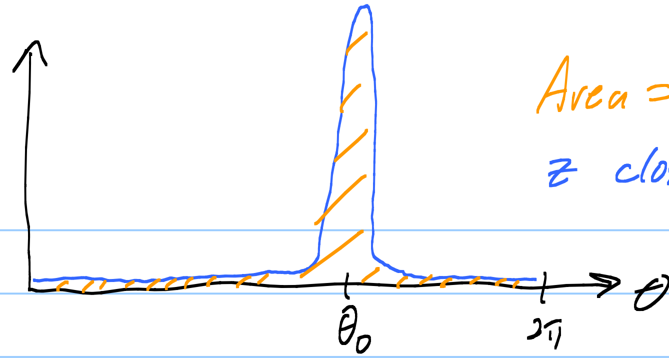
Pf of Schwarz: Saw that $\int_0^{2\pi} P(z,\theta)\varphi(e^{i\theta})\,d\theta = \operatorname{Re} F(z)$ ← analytic

Need to show $u(z) = \int_0^{2\pi} P(z,\theta)\,\varphi(e^{i\theta})\,d\theta \to u(e^{i\theta_0})$ as $z \to e^{i\theta_0}$

Picture of 3:



z not so close to $e^{i\theta_0}$



Area $= 1$

z close to $e^{i\theta_0}$

$$u(z) - \varphi(e^{i\theta_0}) = \int_0^{2\pi} P(z,\theta)\left[\varphi(e^{i\theta}) - \varphi(e^{i\theta_0})\right] d\theta$$

because $\int_0^{2\pi} P \, d\theta = 1$

$$= \int_{I_\delta} + \int_{[0,2\pi]-I_\delta}$$

$$\left|\int_{I_\delta}\right| \le \int_{I_\delta} P(z,\theta)\left|\varphi(e^{i\theta}) - \varphi(e^{i\theta_0})\right| d\theta$$

$|\varphi| \le M$ on $C_1(0)$

$$\le \frac{4}{\pi\delta^2}(1-|z|) \cdot 2M \ (2\pi)$$

if $z \in D_{\delta/2}(e^{i\theta_0})$.

Let $\varepsilon > 0$. φ cont. So $\exists\ \delta > 0$ such that $\left|\varphi(e^{i\theta}) - \varphi(e^{i\theta_0})\right| < \frac{\varepsilon}{2}$ if $|e^{i\theta} - e^{i\theta_0}| < \delta$.

$$\left|\int_{[0,2\pi]-I_\delta}\right| \le \int_{[0,2\pi]-I_\delta} P(z,\theta) \underbrace{\left|\varphi(e^{i\theta}) - \varphi(e^{i\theta_0})\right|}_{< \frac{\varepsilon}{2}} d\theta$$

$$\le \frac{\varepsilon}{2}\int_{[0,2\pi]-I_\delta} P(z,\theta)\, d\theta < \frac{\varepsilon}{2} \underbrace{\int_{[0,2\pi]} P(z,\theta)\, d\theta}_{=1}$$

$$\le \frac{\varepsilon}{2} \ \checkmark$$

Make $\left|\int_{I_\delta}\right| < \frac{\varepsilon}{2}$ by letting $z \to e^{i\theta_0}$ inside $D_{\delta/2}(e^{i\theta_0})$.

Poisson formula for $D_R(z_0)$: $\quad f(z) = R\cdot(z+z_0)$

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, u(z_0 + Re^{i\psi})}{R^2 - 2rR\cos(\theta-\psi) + r^2}\, d\psi$$

Note: $r=0$; See ave prop.

Cor: If $u_n \rightrightarrows u$ are harmonic and converge unif on compact subsets of a domain Ω, then u is harmonic.

Pf #1: Poisson formula.

$$\begin{array}{ccc} u_n & = & \int P(z,\theta)\, u_n \, d\theta \\ \downarrow & & \downarrow \\ u & = & \int P(z,\theta)\, u\, d\theta \quad \checkmark \end{array}$$

Pf #2: u_n continuous. So u continuous.

$\uparrow$

ave prop ✓

Ave prop preserved under unif limits.

u cont satisfies ave prop. So u harmonic.

Weak Averaging Property (WAP): A continuous fcn u on a domain satisfies the WAP if, given a point z_0 in the domain, there is an $\varepsilon > 0$ such that

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta \quad \text{for } 0 < \rho < \varepsilon.$$

Thm: WAP $\Rightarrow$ harmonic

Pf: May assume u cont on $\overline{D_1(0)}$ and satisfies WAP inside.

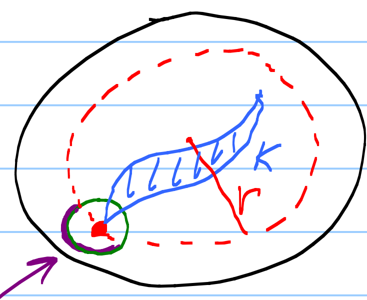
Define $\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta.$ $\leftarrow$ harmonic. $\tilde{u} = u$ on $C_1(0)$,

$\tilde{u}$ satisfies the WAP. Let $U = \tilde{u} - u$. U satisfies the WAP. Suppose $U(w) > 0$ for some $w \in D_1(0)$.

Then $M = \max\limits_{D_1(0)} U$ is > 0.

Note: $K = \{ z \in \overline{D_1(0)} : U(z) = M \}$ is closed and bounded, so compact. And $K \subset D_1(0)$.

Let $r = \max\{ |z| : z \in K \}$.



$U < M$ on more than half of $C_\varepsilon(\bullet)$.

Do WAP at $\bullet \leftarrow w_0$

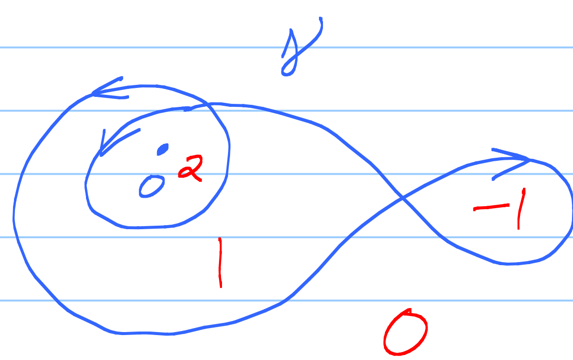
$$M = u(w_0) = \frac{1}{2\pi} \int_{C_\varepsilon(w_0)} u \cdot d\theta < M$$

$\lightning$.

Conclude that $U \le 0$ on $\overline{D_1(0)}$.

Repeat for $u - \tilde{u}$. See $u \equiv \tilde{u}$ $\leftarrow$ harmonic!

HWK 9: Prob. 1:



Hint 1:

Laurent expansion on $\mathbb{C} - \{0\}$.

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\,dw$$

Arg Princ for $f(w) = w - z$

= "winding number"

= # times γ goes around z

$$\int_\gamma f\,dz = \int_\gamma \begin{pmatrix}\text{Laurent} \\ \text{exp}\end{pmatrix} dz$$

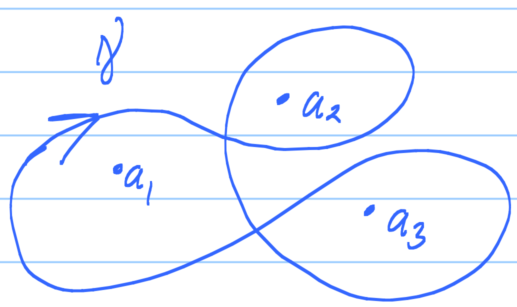
$$= \text{zero} + \int_\gamma \frac{A_{-1}}{z}\,dz + \text{zero}$$

$$= 2\pi i\,\mathrm{Res}_0 f \cdot \underbrace{\mathrm{Ind}_\gamma(0)}_{\text{an integer}}$$

Try

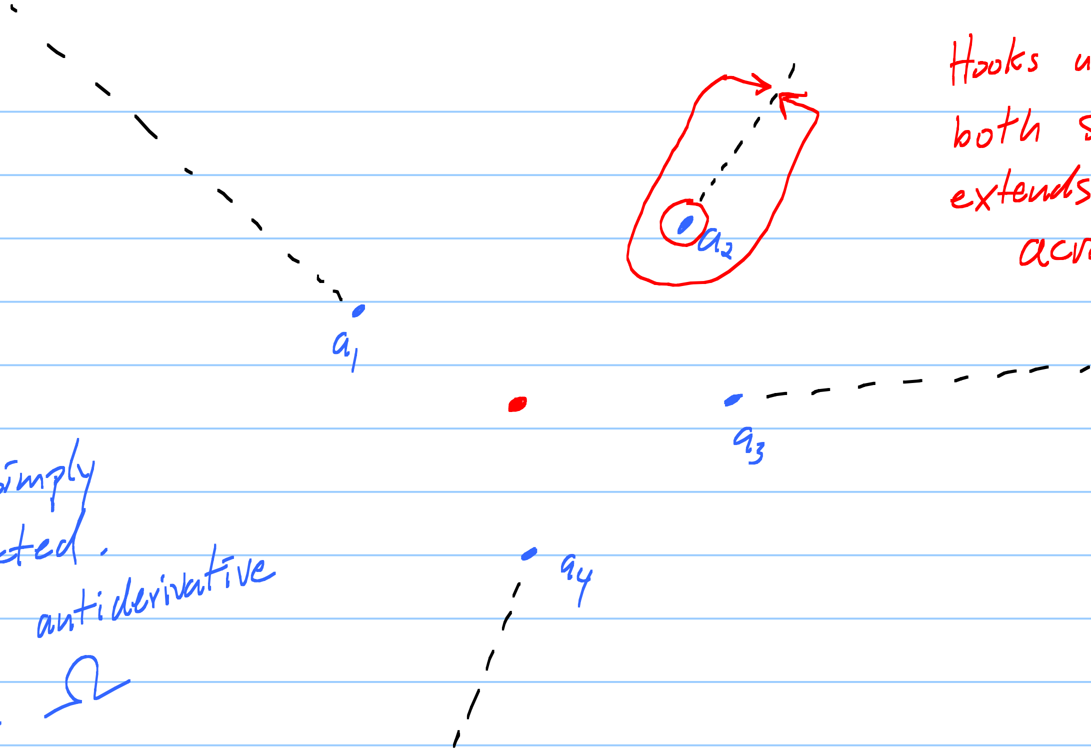
$$F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw\right)$$

Prob 2:



Can use Residue Thm:

$$\int_\gamma f\,dz = 2\pi i \sum \left(\mathrm{Res}_{a_k} f\right) \mathrm{Ind}_\gamma(a_k)$$



Hooks up from both sides.
extends analytically across.

Ω simply connected.
So get antiderivative on Ω