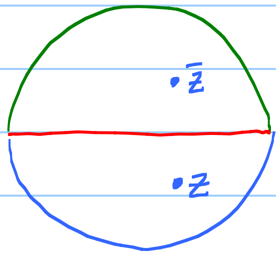
Lecture 35 Schwarz reflection and removable singularities for harmonic fcns.

Reflection Principle for harmonic fcns: Suppose u is continuous on $D_1(0) \cap \{\operatorname{Im} z \geq 0\}$, and $u(x) = 0$ for $x \in \mathbb{R}$.

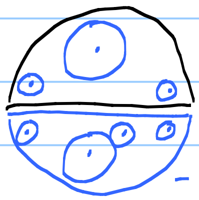


u harmonic on $D_1(0)^+$

Define
$$U(z) = \begin{cases} u(z) & z \in D_1(0) \cap \{\operatorname{Im} z > 0\} \\ 0 & z \in D_1(0) \cap \mathbb{R} \\ -u(\bar{z}) & z \in D_1(0) \cap \{\operatorname{Im} z < 0\} \end{cases}$$

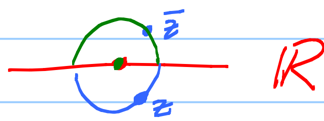
Then U is harmonic on $D_1(0)$.

Pf: U satisfies the WAP.



$-u(\bar{z})$ is harmonic

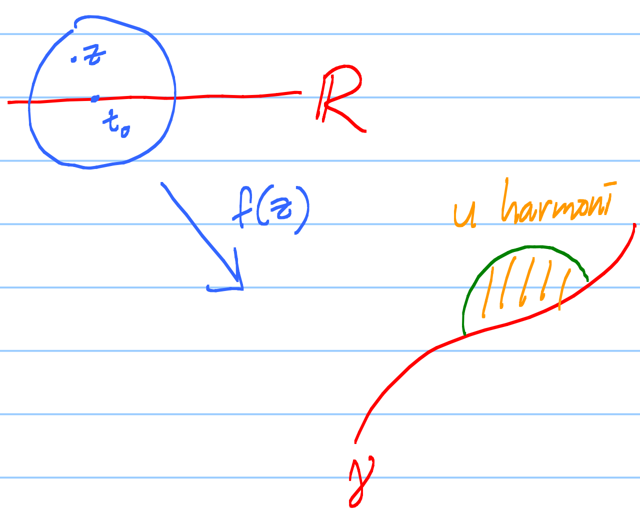
$$\left[u = \operatorname{Re} f \quad u(\bar{z}) = \operatorname{Re} f(\bar{z}) = \operatorname{Re} \overline{F(z)} = \operatorname{Re} F(z) \right]$$



cancelation
$U(z) = -u(\bar{z})$

WAP $\Rightarrow$ U harmonic

Remark: Schwarz reflection get used like this



Real analytic curve:
$z(t) = x(t) + iy(t) \leftarrow$ real power series

Replace t with a z

$[$ assume $z'(t) \neq 0$ $]$

Removable singularity theorem for harmonic fcns

Suppose u is harmonic on $D_r(z_0) - \{z_0\}$ and $|u| \le M$ there. Then u can be given a value at z_0 to make it harmonic on $D_r(z_0)$.

Pf: May assume $r=1$, $z_0 = 0$, u continuous on $\overline{D_1(0)}$, harmonic on $D_1(0)$.

Define $\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$.

Let $\varepsilon > 0$. Define $v_\varepsilon(z) = u(z) - \tilde{u}(z) + \varepsilon \ln|z|$

Note that 1) v_ε harmonic on $D_1(0) - \{0\}$,

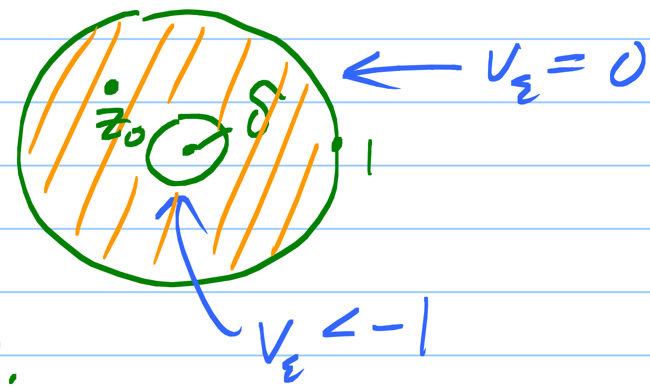
2) v_ε continuous on $\overline{D_1(0)} - \{0\}$,

3) $v_\varepsilon \equiv 0$ on $C_1(0)$, and

4) $\lim_{|z| \to 0} v_\varepsilon(z) = -\infty$. ← Need $|u| \le M$ for this.

Claim: Max Princ $\Rightarrow$ $v_\varepsilon \le 0$ on $D_1(0) - \{0\}$.

Why: Pick $z_0 \in D_1(0) - \{0\}$. Since $v_\varepsilon(z) \to -\infty$ as $z \to 0$, $\exists$ $\delta < |z_0|$ such that $v_\varepsilon(z) < -1$ if $|z| < \delta$, $z \neq 0$.



Max Princ for harmonic fcns $\Rightarrow$ Max v_ε falls on boundary. So $v_\varepsilon(z_0) \leq 0$.

What? $v_\varepsilon = u - \tilde{u} + \varepsilon \operatorname{Ln}|z| \leq 0$

Let $\varepsilon \searrow 0$. See $u - \tilde{u} \leq 0$.

Repeat, using $v_\varepsilon = \tilde{u} - u + \varepsilon \operatorname{Ln}|z|$.

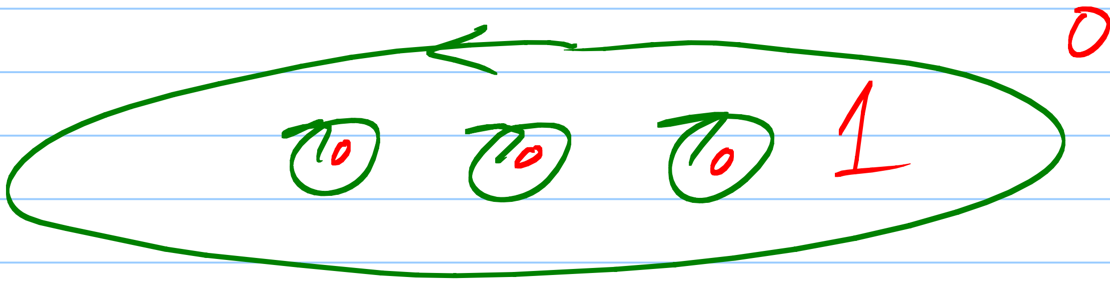
Get $\tilde{u} - u \leq 0$. So $u \equiv \tilde{u}$ ← harmonic at 0.

Define $u(0) = \tilde{u}(0)$.

<u>Def^n</u>: $\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ is called a <u>cycle</u> if each γ_j is a closed curve.

Def^{n}: $$\int_{\Gamma} f \, dz = \sum_{j=1}^{n} \int_{\gamma_j} f \, dz$$

$$\mathrm{Ind}_{\Gamma}(a) = \sum_{j=1}^{n} \mathrm{Ind}_{\gamma_j}(a)$$

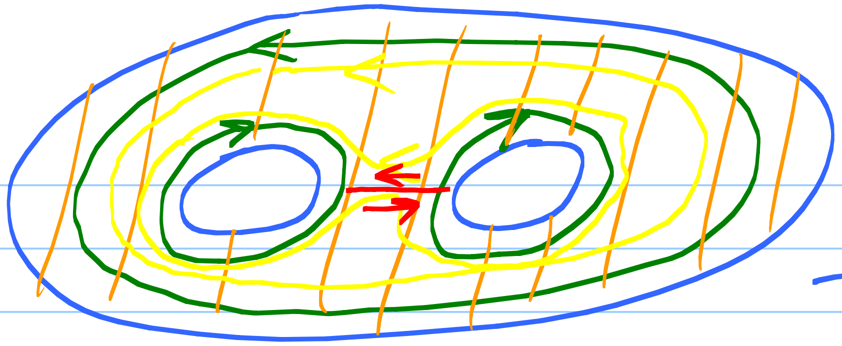


General Cauchy Theorem: Ω a domain, Γ is a cycle in Ω, f analytic on Ω, and $\mathrm{Ind}_{\Gamma}(a) = 0$ for all $a \in \mathbb{C} - \Omega$.

Then 1) $$\int_{\Gamma} f \, dz = 0$$

2) $$\mathrm{Ind}_{\Gamma}(a) f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a} \, dz$$

if $a \in \Omega - tr(\gamma)$.



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