Lecture 4 Complex series

Hwk 1 due Friday

Lemma: If a series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Pf: $S_N = \sum_{n=1}^{N} a_n$. Suppose $S_N \to L$ as $N \to \infty$.

$a_N = S_N - S_{N-1} \longrightarrow L - L$ as $N \to \infty$. ✓

Def^n: A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n| < \infty$.

Lemma: Absolute conv $\Longrightarrow$ convergence.

Power Series: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Bigfact: Power series have a radius of convergence $R \geq 0$, meaning

1) $\sum_0^{\infty} a_n (z-z_0)^n$ converges absolutely for $z \in D_R(z_0)$,
2) converges uniformly on $D_r(z_0)$ for any r with $0 < r < R$.
3) diverges if $|z - z_0| > R$.

Hadamard's Formula: $\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$

Def^n: $\limsup_{n \to \infty} x_n = \lim_{N \to \infty} \Big(\underbrace{\sup\{x_n : n \geq N\}}_{\text{non-increasing seq}} \Big)$

So limit exists. Could be $-\infty$.

Later: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ What is $\limsup_{n \to \infty} \sqrt[n]{n!}$?

Stirling's Formula: $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)$

So $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$.

Take n-th root of first ineq: $\frac{n}{e}(2\pi n)^{\frac{1}{2n}} < \sqrt[n]{n!}$

See $\sqrt[n]{n!} \to \infty$. So $\frac{1}{\sqrt[n]{n!}} \to 0$ as $n\to\infty$,

Conclude: R. of C. of $\sum \frac{z^n}{n!}$ is ∞.

Geometric series: $\sum_{n=0}^{\infty} z^n$.

$S_N = \sum_{n=0}^{N} z^n$

$$\begin{aligned} S_N &= 1 + z + \cdots + z^N \\ - \quad z S_N &= \phantom{1 + {}} z + \cdots + z^N + z^{N+1} \\ \hline (1-z)S_N &= 1 - z^{N+1} \end{aligned}$$

So $S_N = \underset{\substack{\uparrow \\ \text{limit}}}{\frac{1}{1-z}} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$

$|z+w| \le |z| + |w| \leftarrow$ numerator estimate

$|z-w| \ge \big||z|-|w|\big| \leftarrow$ denominator estimate

or $|z+w| \ge \big||z|-|w|\big|$

Suppose $|z| < r < 1$. $|E_N(z)| = \frac{|z^{N+1}|}{|1-z|} \le \frac{|z^{N+1}|}{\big|1-|z|\big|}$

$= \frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-r} \longrightarrow 0$ as $N\to\infty$,

So $\sum z^n \to \frac{1}{1-z}$ uniformly on $D_r(0)$ when $0<r<1$.

Note: If $|z|>1$, then $|z^n| = |z|^n > 1$.

So terms $\not\to 0$. $\sum z^n$ diverges. R. of C $= 1$.

Note: $E_N(x) = \dfrac{x^{N+1}}{1-x} \to \infty$ as $x \nearrow 1$. So no uniform conv on $D_1(0)$.

Pf of Hadamard: Case $R \neq 0$, $R \neq \infty$. Can assume $z_0 = 0$ via change of var $w = z - z_0$. Suppose $0 < r < R$.

Pick ρ with $0 < r < \rho < R$. Take z with

$|z| < r < \rho < R$.

Then $\dfrac{1}{R} < \dfrac{1}{\rho}$ ← $\sup\limits_{n \geq N} \sqrt[n]{|a_n|}$ sliding down to $\frac{1}{R}$

So $\exists\, N_0$ such that $\sup\limits_{n \geq N_0} \sqrt[n]{|a_n|} < \dfrac{1}{\rho}$

So $\sqrt[n]{|a_n|} < \dfrac{1}{\rho}$ for $n \geq N_0$

n-th power: $|a_n| < \left(\dfrac{1}{\rho}\right)^n$, $n \geq N_0$

Next $|a_n z^n| < \left(\dfrac{|z|}{\rho}\right)^n < \left(\dfrac{r}{\rho}\right)^n$

Aha! $a_n z^n$ comparable to $\left(\dfrac{r}{\rho}\right)^n$ for $n \geq N_0$.

So $\sum a_n z^n$ converges absolutely ↖ < 1

Next: Get unif conv on $D_r(0)$. Let $f(z) = \sum\limits_0^\infty a_n z^n$.

N_0 same as above. $\left| f(z) - \sum\limits_0^N a_n z^n \right| = \left| \sum\limits_{N+1}^\infty a_n z^n \right|$

$$\le \sum_{N+1}^{\infty} |a_n z^n| < \underbrace{\sum_{N+1}^{\infty} \left(\frac{r}{\rho}\right)^n}_{\frac{\left(\frac{r}{\rho}\right)^{N+1}}{1-\frac{r}{\rho}}} \leftarrow \text{if } N > N_0$$

$$\frac{\left(\frac{r}{\rho}\right)^{N+1}}{1-\frac{r}{\rho}} \to 0 \text{ as } N \to \infty,$$

Last part: Divergence $\underbrace{|z| > R}_{\frac{1}{|z|} < \frac{1}{R}}$.

$\frac{1}{|z|} < \frac{1}{R} \leftarrow \sup_{n \ge N} \sqrt[n]{|a_n|}$ sliding down to $\frac{1}{R}$

Conclude: For each N, there is a $n > N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$\frac{1}{|z|^n} < |a_n|$$

$$1 < |a_n z^n| \leftarrow \text{Aha! Terms don't go to zero!}$$

Remark: $R=0$, $R=\infty$ exercise.

EX: $\sum_{n=0}^{\infty} n!\, z^n$ only converges at $z=0$. $R=0$.

Ratio Test: $\sum_{n=0}^{\infty} c_n$ Suppose $\frac{c_{n+1}}{c_n} \to L$ as $n \to \infty$.

If $|L| < 1$, series converges. (compare to geom ser.)

$|L| > 1$, series diverges (terms $\not\to 0$)

$|L| = 1$, test fails.

Exercise: $\{a_n\}_{n=1}^{\infty}$ be an enumeration of all points in $D_1(0)$ with rational real and imag parts. Show $\sum_{n=1}^{\infty} a_n z^n$ has R. of C. $= 1$.