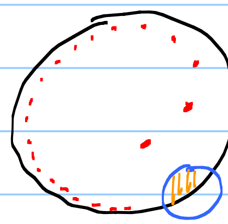
Lecture 42 Review 1

Fact: Every domain in $\mathbb{C}$ is a "domain of holomorphy," meaning I can find analytic fcn on the domain that can't be continued to be analytic on any bigger domain.

EX: $D_1(0)$



Zeroes by Weierstraß Thm that spiral out to the boundary and every bndry pt is a limit pt.

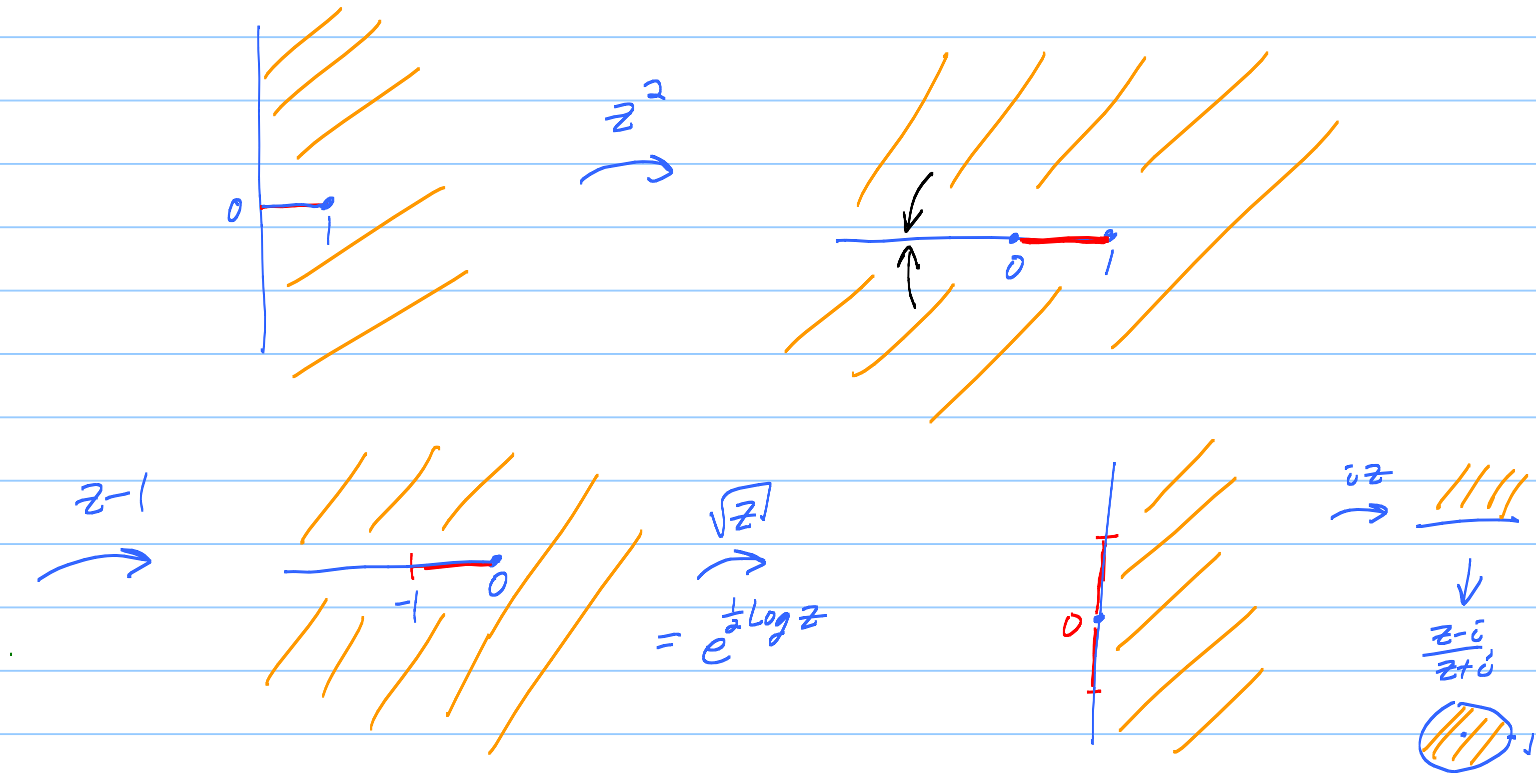
Fact: in $\mathbb{C}^n$, $n>1$, analytic fcns cannot have any isolated singularities! Ball $-\{0\}$ is not a domain of holomorphy.

$D_1(0)$ example above:
$$B(z) = \prod_{n=1}^{\infty} \bar{a}_n \frac{(a_n - z)}{1-\bar{a}_n z}$$

converges unif on compacts to an analytic fcn with zeroes at $a_n \in D_1(0)$ when $\sum_{n=1}^{\infty} (1-|a_n|) < \infty$.

Practice problems

3. *(30 pts.)* Find a one-to-one conformal mapping from the region $\{z : \operatorname{Re} z > 0\} - (0, 1]$ onto the unit disc.



4. *(30 pts.)* Suppose that $f_n(z)$ is a sequence of functions that are continuous on $\overline{D_1(0)}$, analytic on $D_1(0)$, and such that $\int_0^{2\pi} |f_n(e^{i\theta})|\, d\theta < 1$ for all n. Prove that there is a subsequence that converges uniformly on compact subsets of $D_1(0)$.

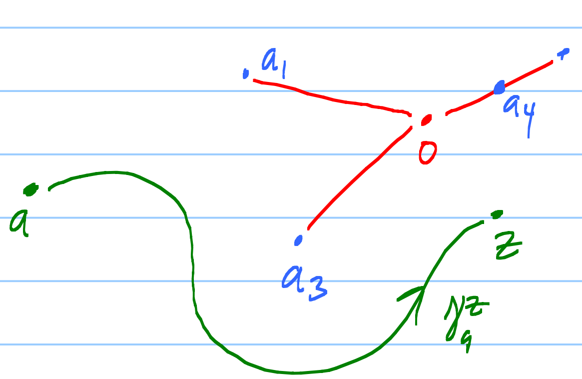
Suppose $K \subset D_1(0)$ is compact. Then $K \subset D_\rho(0)$ for some ρ with $0<\rho<1$.

$$|f_n(z)| = \left|\frac{1}{2\pi i}\int_{C_1(0)} \frac{f_n(w)}{w-z}\,dw\right| = \left(\left|\lim_{r\nearrow 1}\frac{1}{2\pi i}\int_{C_r(0)}\frac{f_n(w)}{w-z}\,dw\right|\right)$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \overbrace{\frac{1}{|e^{i\theta}-z|}}^{\le\frac{1}{1-\rho}} \cdot |f_n(e^{i\theta})|\,|ie^{i\theta}|\,d\theta \le \frac{1}{2\pi(1-\rho)}\int_0^{2\pi}|f_n|\,d\theta.$$ Montel's!

5. *(40 pts.)* Suppose $a_1, a_2, \ldots, a_N$ are distinct non-zero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^{N} (z - a_k).$$



$\Omega = \mathbb{C} - \text{Lines}$

Let $F(z) = \prod_{k=1}^{N} (z - a_n)$.

Hmmm. $f(z) \overset{?}{=} \exp\left(\frac{1}{N} \int_{\gamma_a^z} \frac{F'(w)}{F(w)}\, dw \right)$

Step 1: $f(z)$ is well defined.

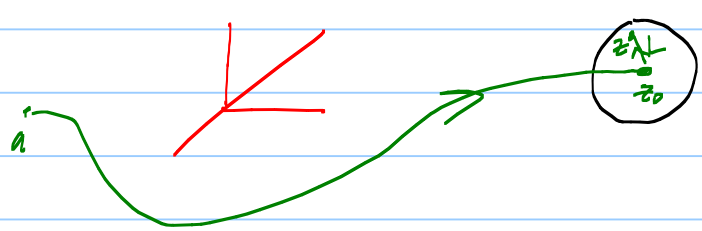
$$\frac{f(z)}{\tilde{f}(z)} = \exp\left(\frac{1}{N} \underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z} \right)}_{\text{closed in } \Omega} \frac{F'(w)}{F(w)}\, dw \right)$$

$$= \exp\left(\frac{1}{N} 2\pi i \sum_{n=1}^{N} \underbrace{\operatorname{Ind}_\Gamma a_n}_{\substack{\text{all the same, say} = m \\ \text{because } a_n\text{'s in} \\ \text{same component of } \mathbb{C} - \Gamma}} \underbrace{\operatorname{Res}_{a_n} \frac{F'}{F}}_{\substack{=1 \\ \text{at simple zero}}} \right)$$

Residue Thm on $\mathbb{C}$.

$$= e^{\frac{1}{N} 2\pi i\, m \cdot N} = e^{2\pi i m} = 1. \checkmark$$

Step 2: f is analytic and $f'(z) = f(z) \frac{1}{N} \frac{F'(z)}{F(z)}$. $\checkmark$

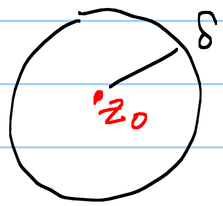


$$f(z) = \exp\left(c + \underbrace{\frac{1}{N} \int_{z_0}^{z} \frac{F'}{F}\, dw}_{\text{antider for } \frac{F'}{F} \text{ on disc}} \right)$$

Step 3: $\frac{f(z)}{F(z)^N}$ ← deriv $\equiv 0$, so const.

Step 4: Correct by constant multiple.

6. (*40 pts.*) Suppose that $f(z)$ is an analytic functions with a zero of order N at z_0. Prove that there exist $\epsilon > 0$ and $\delta > 0$ such that, for every $w \in \mathbb{C}$ with $0 < |w| < \epsilon$, the equation $f(z) = w$ has exactly N *distinct* roots in $D_\delta(z_0)$.
Hint: Rouché's Theorem.



Shrink δ so that z_0 is <u>only</u> zero of f on $\overline{D_\delta(z_0)}$. Rouché: $f(z) = f(z)$

$g(z) = f(z) - w$

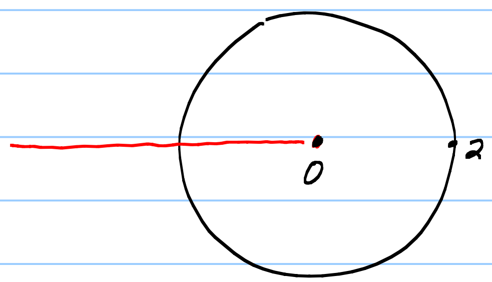
$$|f(z) - g(z)| = |w| \underset{\uparrow \text{ want}}{<} \underbrace{|f(z)|}_{\text{has a positive min } m \text{ on } C_\delta(z_0).} \text{ on } C_\delta(z_0)$$

Aha! Need $0 < \varepsilon < m$.

Rouché gives N zeroes of $f(z) - w$ inside $C_\delta(z_0)$ counted with <u>multiplicity</u>.

Aha! Only get multiple zeroes at places where $\frac{d}{dz}[f(z) - w] = f'(z)$ vanishes. $f'(z_0) = 0$. Shrink δ more perhaps so that z_0 is the only zero of f' on $\overline{D_\delta(z_0)}$.

7. Prove that $\ln|z|$ does not have a harmonic conjugate on $\{z : 1 < |z| < 2\}$.



$$\ln|z| = \text{Re} \underbrace{\text{Log}}_{\text{Princ. Branch}} z \text{ on } \underbrace{D_2(0) - (-2, 0]}_{\text{simply connected}}$$

Hmmm. Suppose v is a harm conj for $\ln|z|$ on $D_2(0) - \{0\}$. It is also a harm conj on $D_2(0) - (-2, 0]$.

So $v = \text{Arg}\, z + C$ on $D_2(0) - (-2, 0]$.

Oops! See that v is not continuous across the cut! ↯.