Lecture 6 Goursat's Lemma

$\gamma : z(t)$
$a \le t \le b$

Basic Estimate: $\left| \int_\gamma f\, dz \right| \le M \cdot \text{Length}(\gamma)$

where $M = \underset{z \in tr(\gamma)}{MAX} |f(z)|$, $\text{Length}(\gamma) = \int_a^b |z'(t)|\, dt$

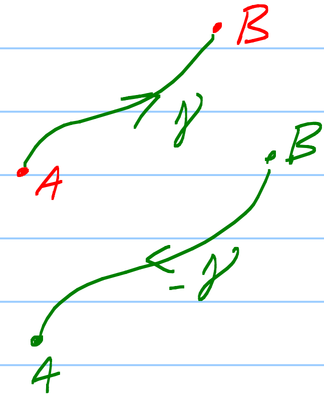
$$= \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

Pf: $\left| \int_a^b f(z(t)) \cdot z'(t)\, dt \right|$

$$\le \int_a^b \underbrace{|f(z(t))|}_{\le M} \cdot |z'(t)|\, dt \le M \int_a^b |z'(t)| dt \quad \checkmark$$

Def^n: $\gamma : z(t)$, $a \le t \le b$

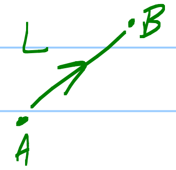
$-\gamma : z(b + t(a-b))$, $0 \le t \le 1$



Fact: $\int_{-\gamma} f\, dz = - \int_\gamma f\, dz$

Pf: Write out from def^n, separate real and imag parts, use change of vars formula.

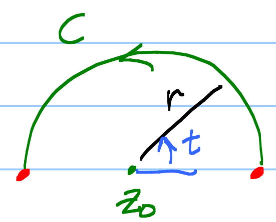
EX:



$L : z(t) = A + t(B - A)$, $0 \le t \le 1$

$z'(t) = B - A$

EX:



$C : z(t) = z_0 + re^{it}$, $0 \le t \le \pi$

$$z'(t) = 0 + r \underbrace{\frac{d}{dt}\left[E(it)\right]}_{E'(it) \cdot \frac{d}{dt}(it) = i E(it)} = ire^{it}$$

Facts: $\frac{d}{dz}(z^n) = n z^{n-1}$

Polynomials are analytic

Polynomials have polynomial antiderivatives. $\frac{d}{dz}\left[\frac{1}{n+1} z^{n+1}\right] = z^n$

<u>Consequence</u>: $\int_\gamma P(z)\,dz = \int_\gamma Q'(z)\,dz = Q(\text{END}) - Q(\text{START})$

$P(z)$: poly. $Q'(z)$: poly anti-der.

$= 0$ if γ <u>closed</u>.

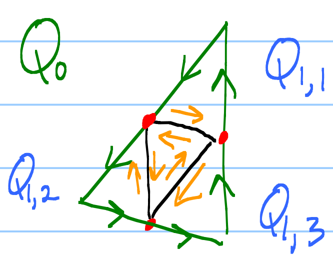
<u>and</u> $\int_\gamma az+b\,dz = 0$ for closed γ.

<u>Goursat's Lemma</u>: $\Omega \subset \mathbb{C}$ open, $p \in \Omega$, f continuous on Ω and analytic on $\Omega - \{p\}$, and Q is a triangle in Ω, $\bar{Q} \subset \Omega$. Then $\int_Q f\,dz = 0$.

<u>Pf</u>: Case p outside Q. Write $Q_0 = Q$.

<u>Baby Lemma</u>; If $a_1 + a_2 + a_3 + a_4 = I$, then $\exists j$ such that $|a_j| \geq \frac{|I|}{4}$.

<u>Pf</u>: $I = 0$ case. Duh. $I \neq 0$. Δ ineq.



Aha! $I = \int_{Q_0} f\,dz = \sum_{j=1}^{4} \int_{Q_{1,j}} f\,dz$

$Q_{1,4}$: middle one, counterclockwise

Lemma $\Rightarrow \exists\, j_1$ with

$$\left|\int_{Q_{1,j_1}} f\,dz\right| \geq \frac{|I|}{4}.$$

Write $Q_1 = Q_{1,j_1}$

Repeat! Cut up Q_1 into $Q_{2,1}, Q_{2,2}, \ldots, Q_{2,4}$.

Get Q_{2,j_2} with $\left|\int_{Q_{2,j_2}} f\,dz\right| \geq \frac{|I|}{4^2}$

Continue. Get $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$ with

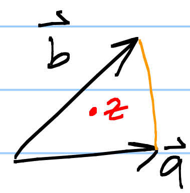
$$\left| \int_{Q_n} f \, dz \right| \geq \frac{|I|}{4^n}$$

<u>Topology Lemma</u>: Suppose $K_n \neq \emptyset$ are compact in $\mathbb{R}^2$ and $K_0 \supset K_1 \supset K_2 \supset \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Furthermore, if $\operatorname{Diam}(K_n) \to 0$ as $n \to \infty$, then

$\bigcap K_n = \{z_0\}$, a single pt.

<u>Δ pf</u>:



$z = x\vec{a} + y\vec{b}$, x, y "coords"

Take the lower left pt of Q_n's. Get non-decreasing seq's x_n, y_n bounded below by 0. They converge! $z_0 = x_0\vec{a} + y_0\vec{b}$.

Back to Goursat. $\bigcap Q_n = \{z_0\}$. Let $\varepsilon > 0$.

$$\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + E(z)$$ where $E(z) \to 0$ as $z \to z_0$.

Define $E(z_0) = 0$, cont.

$$\boxed{f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\text{linear}} + E(z)(z - z_0)}$$

holds at z_0 too.

f complex diff'ble at z_0, so $\exists\ \delta > 0$ such that $|E(z)| < \varepsilon$ when $z \in D_\delta(z_0)$. $\exists\ N$ such that $Q_n \subset D_\delta(z_0)$ for $n > N$.

$$\int_{Q_n} f \, dz = \underbrace{\int_{Q_n} (\text{Complex linear}) \, dz}_{=0} + \int_{Q_n} E(z)(z - z_0) \, dz$$

$$\frac{|I|}{4^n} \le \left| \int_{Q_n} f\, dz \right| = \left| \int_{Q_n} E(z)(z-z_0)\, dz \right|$$

$$\le \Big(\max_{z \in Q_n} \underbrace{|E(z)|}_{\le \varepsilon}\, \underbrace{|z-z_0|}_{\le \text{Diam}} \Big) \big(\text{Length}(Q_n)\big)$$

$n > N$

$Q_n \subset D_\delta(z_0)$

$$\le \varepsilon \cdot \underbrace{\text{Diam}(Q_n)}_{=\frac{1}{2^n}\text{Diam}(Q_0)} \cdot \underbrace{\text{Length}(Q_n)}_{=\frac{1}{2^n}\text{Diam}(Q_0)}$$

Aha! $\frac{|I|}{4^n} \le \frac{\varepsilon}{4^n} \cdot C$ where $C = \text{Diam}(Q)\,\text{Length}(Q)$.

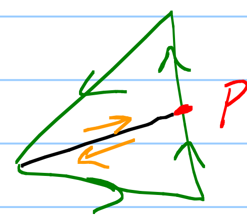
$|I| \le C \cdot \varepsilon$ where $\varepsilon > 0$ arbitrary! So $I = 0$. ✓

Case p inside Q:

$$\int_Q = \sum_{\text{Three}} \int_{Q_j}$$

Reduced to p = vertex case.

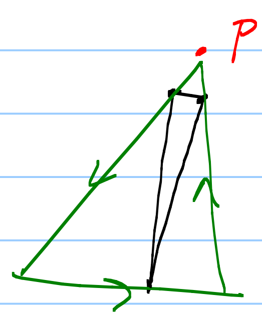
Case p on a side:



$$\int_Q = \sum_{\text{Two}} \int Q_j$$

Reduced to p = vertex.

Case p = vertex:



$$\int_Q = \int_{\substack{\text{Tiny}\\ Q \text{ at}\\ \text{top}}} + \underbrace{\sum_{\text{THREE}} \int_{Q_j}}_{=0}$$

But $\left| \int_{\text{Tiny } Q} \right| \le \Big(\max_{D_\delta(p)} |f| \Big) \cdot \text{Length}(Q_{\text{Tiny}})$ ← can be made arb. small.

↑ fix δ.

So $\int_Q f\, dz = 0$. Done!