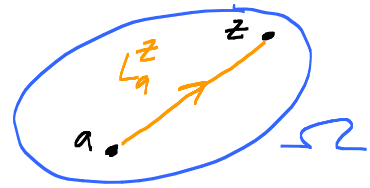
Lecture 7 Cauchy Theorem on a convex open set



Suppose Ω is a convex open set in $\mathbb{C}$ and suppose f is continuous on Ω and analytic on $\Omega - \{p\}$. Then $\int_\gamma f\,dz = 0$ for any closed path in Ω.

Proof: Plan: Cook up an F so that $F' = f$. ← continuous

F. Thm. Calc #2 $\Longrightarrow$ $\int_\gamma f\,dz = \int_\gamma F'\,dz = F(\text{END}) - F(\text{START}) = 0$ if γ closed.

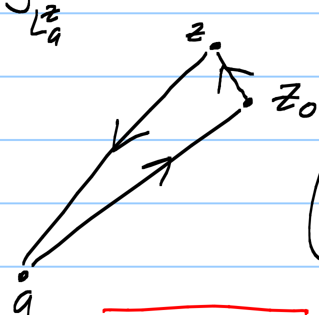
Big idea: If we had such an F. Then

$$\int_{L_a^z} f(w)\,dw = \int_{L_a^z} F'(w)\,dw = F(z) - \underbrace{F(a)}_{\text{const.}}$$

Aha! $$\frac{d}{dz}\Big(\underbrace{\int_{L_a^z} f(w)\,dw}_{\text{anti der for } f}\Big) = F'(w) = f(w)$$

Big idea: Define $F(z) = \int_{L_a^z} f(w)\,dw$. Show $F' = f$.

Key: Goursat



$$\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \int_{-L_a^z}\right) f\,dw = 0$$

$\uparrow = -\int_{L_a^z}$

$$\boxed{\int_{L_a^z} f\,dw - \int_{L_a^{z_0}} f\,dw = \int_{L_{z_0}^z} f\,dw}$$

$\uparrow$ $\uparrow$

$$DQ = \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\int_{L_{z_0}^z} f\,dw$$

Little fact: $\int_\gamma c\,dw = \int_\gamma \frac{d}{dw}[cw]\,dw = c\,[(\text{END}) - (\text{START})]$

So $\displaystyle\int_{L_{z_0}^{z}} c\, dw = c(z-z_0)$.

Let $\varepsilon>0$. f cont on Ω, so $\exists\, \delta>0$ such that $|f(z)-f(z_0)|<\varepsilon$ if $z\in D_\delta(z_0)\subset\Omega$. Write $f(z)=f(z_0)+E(z)$ where $|E(z)|<\varepsilon$ on $D_\delta(z_0)$.

$$DQ = \frac{1}{z-z_0}\left[\underbrace{\int_{L_{z_0}^{z}} f(z_0)\,dw}_{f(z_0)(z-z_0)} + \int_{L_{z_0}^{z}} E(w)\,dw\right]$$

$$DQ = f(z_0) + \underbrace{\frac{1}{z-z_0}\int_{L_{z_0}^{z}} E(w)\,dw}_{\mathcal{E}(z)}$$

Now $\displaystyle |\mathcal{E}(z)| = \frac{1}{|z-z_0|}\left|\int_{L_{z_0}^{z}} E(w)\,dw\right| \le \frac{1}{|z-z_0|}\underbrace{\max_{L_{z_0}^{z}}|E(w)|}_{\le\varepsilon}\cdot\underbrace{\text{Length}(L_{z_0}^{z})}_{|z-z_0|}$

$\le \varepsilon$ if $|z-z_0|<\delta$.

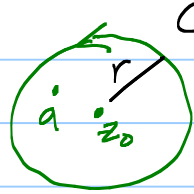
Conclude F is analytic on Ω and $F'=f$.

Future: Replace convex open by simply connected open

Cor: Analytic fcns on a convex open have analytic antiderivs.

Next stop: Cauchy Integral Thm on a disc.

Lemma:



$C_r(z_0)$: $z(t)=z_0+re^{it}$, $0\le t\le 2\pi$

$$\int_{C_r(z_0)} \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & \text{if } a\in D_r(z_0) \\ 0 & \text{if } |a-z_0|>r \end{cases}$$

Pf: 0 case is Cauchy on convex for $\Omega = D_{|a-z_0|}(z_0)$ ← convex open

$\frac{1}{z-a}$ analytic on Ω.

Case $a=z_0$: $\displaystyle\int_{C_r(z_0)} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{\underbrace{(z_0+re^{it})}_{z(t)}-z_0}\underbrace{[ire^{it}]}_{z'(t)}\,dt$

$$= \int_0^{2\pi} i \, dt = 2\pi i$$

<u>Case $a \in D_r(z_0)$:</u> Define $\displaystyle H(w) = \int_{C_r(z_0)} \frac{1}{z-w}\, dz.$

HWK 2; Problem 1:

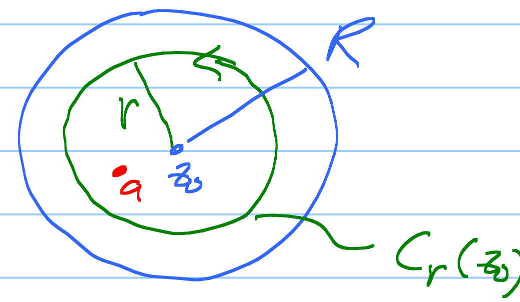
$$H'(w) = \int_{C_r(z_0)} \frac{1}{(z-w)^2}\, dz$$

$$= \int_{C_r(z_0)} \frac{d}{dz}\left[\frac{-1}{(z-w)}\right] dz \equiv 0$$

So $H(w)$ is const.
But $H(z_0) = 2\pi i$, So $H(w) \equiv 2\pi i$

<u>Cauchy Int. formula on a disc.</u>

f analytic on $D_R(z_0)$, $0<r<R$, $a \in D_r(z_0)$.



$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz$$

<u>Pf</u>; $D_R(z_0)$ is convex, open. (Think $p=a$)

$$G(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

G is continuos, analytic on $D_R(z_0) - \{a\}$,

Cauchy on convex $\Rightarrow \displaystyle\int_{C_r(z_0)} G(z)\, dz = 0$

$$\int_{C_r(z_0)} \frac{f(z)-f(a)}{z-a}\, dz = 0$$

$$\int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz = \int_{C_r(z_0)} \frac{f(a)}{z-a}\, dz = f(a) \underbrace{\int_{C_r(z_0)} \frac{1}{z-a}\, dz}_{=2\pi i}$$

(Lemma)

Big consequence: f analytic $\Rightarrow$ f' analytic

Why:
$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$$

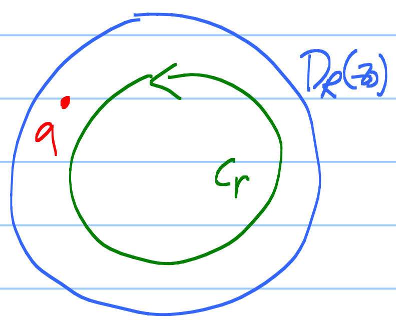
HWK 2: #1:
$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw$$

$$f''(z) = \frac{2}{2\pi i} \int_C \frac{f(w)}{(w-z)^3}\, dw$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw \quad \leftarrow \text{Cauchy's Formula}$$

Remark:



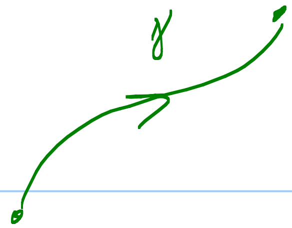
$$\int_{C_r} \frac{f(z)}{\underbrace{z-a}_{\text{analytic inside}}}\, dz = 0 \quad \text{by Cauchy on convex}$$

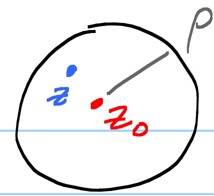
HWK 2: #1:
$$f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw$$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{z-z_0} \int_\gamma \varphi(w) \left[\underbrace{\frac{1}{w-z} - \frac{1}{w-z_0}}_{\frac{z-z_0}{(w-z)(w-z_0)}} \right] dw$$

$$= \int_\gamma \frac{\varphi(w)}{(w-z)(w-z_0)}\, dw \xrightarrow{\text{Guess!}} \underbrace{\int_\gamma \frac{\varphi(w)}{(w-z_0)^2}\, dw}_{I}$$

Prove it!
$$|DQ - I| = \underset{\to 0}{|z-z_0|} \left| \int_\gamma \frac{\varphi(w)}{\text{poly}}\, dw \right| \leftarrow \text{show bounded}$$





$$\rho = \frac{1}{2} \underbrace{dist(z_0, \gamma)}_{\inf_{a \le t \le b} |z_0 - z(t)|}$$