Lecture 9 Analytic fcns are = convergent power series (HWK 3 due Mon.)

Cauchy Thm on Convex Open Ω: f analytic $\Rightarrow \int_\gamma f\,dz = 0$ $\forall$ closed γ.

Stronger converse: Ω open, f cont, $\int_{b\Delta} f\,dz = 0$ $\forall$ Δ's $\subset \Omega \Rightarrow f$ analytic

(Morera's Thm)

Warning: EX: $\Omega = \{z : r < |z| < R\}$, $f(z) = \frac{1}{z}$

Take ρ with $r < \rho < R$. $\int_{C_\rho(0)} \frac{1}{z}\,dz = 2\pi i \neq 0$

Later: Cauchy Thm holds for Ω simply connected domains.

$$\underbrace{\text{simply connected}}_{\text{no holes}} \qquad \underbrace{\text{domains}}_{\text{connected open}}$$

Notation: $S_N(z) = 1 + z + \cdots + z^N$

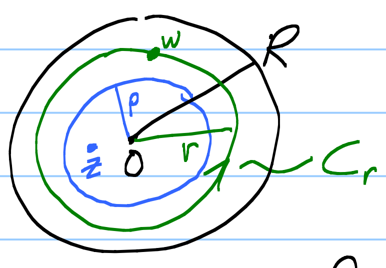
$$\frac{1}{1-z} = S_N(z) + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

If $|z| < \rho < 1$, then

$$|E_N(z)| < \frac{\rho^{N+1}}{1-\rho}.$$

Thm: Suppose f analytic on $D_R(z_0)$. Then f is given by a convergent power series $f(z) = \sum_0^\infty a_n (z-z_0)$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$ and R. of C. is $\geq R$.

Pf: Can assume $z_0 = 0$ via change of var $\zeta = z - z_0$.



Let $0 < \rho < r < R$. Then

$$\left|\frac{z}{w}\right| = \frac{|z|}{r} < \left(\frac{\rho}{r}\right) < 1$$

$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\,\frac{1}{1-\frac{z}{w}}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\left[1 + \left(\frac{z}{w}\right) + \cdots + \left(\frac{z}{w}\right)^N\right] dw + \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w} E_N\!\left(\frac{z}{w}\right) dw}_{\mathcal{E}_N(z)}$$

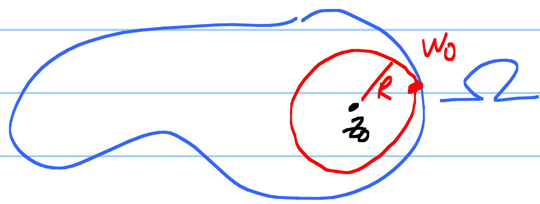
$$= \underbrace{\sum_{n=0}^{N} \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{n+1}}\, dw \right)}_{\text{Cauchy Formula! } a=0 \; = \frac{f^{(n)}(0)}{n!}} z^n \; + \; \mathcal{E}_N(z)$$

$$|\mathcal{E}_N(z)| \le \frac{1}{2\pi} \frac{\max_{C_r} |f|}{r} \cdot \frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}} \cdot 2\pi r \quad \left(\tfrac{\rho}{r} < 1\right)$$

$\to 0$ as $N \to \infty$.

Power series converges uniformly on $D_\rho(0)$. Can make ρ as close to R as desired. So R. of C is $\ge R$.

Consequence: f anal on Ω, R. of C $\ge R$.



$D_R(z_0) = \bigcup D_\rho(z_0)$

$D_\rho(z_0) \subset \Omega$

$f(z) = \frac{1}{z - w_0}$ shows R. of C might $= R$.

Today: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ with R of C $= \infty$.

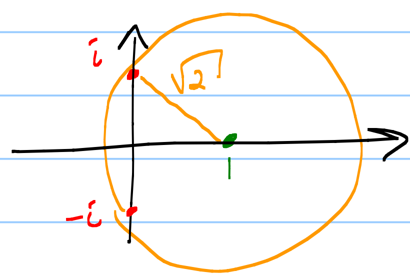
$\left[\begin{array}{l} E'(z) = E(z) \\ \text{entire} \\ E(0) = 1 \end{array} \right.$

Def^n: $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ $\quad R = \infty$

$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$ $\quad R = \infty$

$\tan z = \frac{\sin z}{\cos z}$ $\quad R = ?$

Ex: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$. $\quad \underbrace{f(z) = \sum_0^\infty a_n (z-1)^n}$ $\leftarrow R \ge \sqrt{2}$



if converges on $D_r(1)$ where $r > \sqrt{2}$, then it converges to $F(z)$ analytic on $D_r(1)$. But $f \equiv F$ on $D_{\sqrt{2}}(1)$

blows up at i $\quad$ doesn't!

So $R = \sqrt{2}$.

Thm: Suppose $f(z) = \sum_0^\infty a_n (z-z_0)^n$ with R of C $R_f > 0$.

We know f anal on $D_{R_f}(z_0)$. So f' is anal there.

So $f' = \sum_{n=0}^\infty b_n (z-z_0)^n$ with $R_{f'} \geq R_f$.

In fact, $R_{f'} = R_f$ and $b_{n-1} = n\, a_n$, i.e., we can diff'tiate p. series term by term.

Pf: $b_{n-1} = \dfrac{(f')^{(n-1)}(z_0)}{(n-1)!} = \dfrac{n f^{(n)}(z_0)}{\underbrace{n\cdot(n-1)!}_{n!}} = n\, a_n$ ✓

Know $R_{f'} \geq R_f$. If $R_{f'} > R_f$, then $\sum b_n (z-z_0)^n \to g(z)$ analytic on bigger disc. Let $G(z)$ be an antider of g there. Then $G' = f'$

$\frac{d}{dz}[G-f] \equiv 0$. So $G - f = c$

$f = G - c \leftarrow$ has R of C $> R_f$!

Note: Hadamard proof: $\limsup_{n\to\infty} \sqrt[n-1]{n|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

True: $(\sqrt[n]{n} \to 1)$.

Zeroes of analytic fcns: (polynomialesque.)

Suppose f analytic on $D_R(z_0)$ and $f(z_0) = 0$.

Then $f(z) = \underbrace{\sum_{n=0}^\infty a_n (z-z_0)^n}_{\text{R. of C} \geq R}$, $\left[a_0 = \frac{f(z_0)}{0!} = 0 \right]$

Case: All a_n's $= 0$. Then $f(z) \equiv 0$ on $D_R(z_0)$.

Case: Not all zero. Let N be smallest n such that $a_n \neq 0$.

Def^n: N is the multiplicity of the zero at z_0.

$$f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$

$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots \right]}$$

converges for exactly same z as series for f. So same R. of C. $\geq R$

Converges to anal fcn $H(z)$.

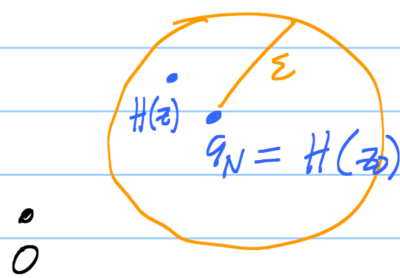
Note: $H(z_0) = a_N \neq 0$.

$$f(z) = (z-z_0)^N H(z)$$

Aha! Zeroes of f are isolated! Just like polys.

Analysis lemma: H analytic $\Longrightarrow$ H continuous.

$H(z_0) \neq 0$. So $\exists$ $\delta > 0$ such that $H(z) \neq 0$ if $|z-z_0| < \delta$.



$\varepsilon = \frac{|a_N|}{2}$

δ from def^n of continuity