Lecture 3 C-R Eqns and the complex exponential

Fact: u $\mathcal{C}^1$-smooth :

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + R(x,y)$$

where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

Thm: u,v $\mathcal{C}^1$-smooth & satisfy C-R Eqns. Then $f=u+iv$ analytic.

Pf:

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0} \qquad z_0 = x_0+iy_0, \quad z = x+iy$$

$$= \frac{[u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0)] + i[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)]}{z-z_0} + \frac{R_u+iR_v}{z-z_0}$$

$$= \frac{[u_x^0+iv_x^0]\cancel{((x-x_0)+i(y-y_0))}}{\cancel{z-z_0}} + \frac{R}{z-z_0}$$

$$= (u_x^0+iv_x^0) + \mathcal{E}$$

where $\displaystyle |\mathcal{E}| \le \frac{|R_u+iR_v|}{|z-z_0|} \le \frac{|R_u|}{|z-z_0|} + \frac{|R_v|}{|z-z_0|} \to 0$ as $z\to z_0$

Cor: $E(z)$ is analytic on $\mathbb{C}$ $\leftarrow$ it is entire.

$$E(x+iy) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

Properties of $E(z)$: 0) $E(z)$ is entire

1) $E'(z) = u_x + i v_x = u + iv = E(z)$

2) $E(0) = 1$

0–2 determine $E(z)$

3) $E(z+w) = E(z)E(w)$

4) $E(-z) = 1/E(z)$

5) $E(z)$ is never zero

6) $E(z-w) = E(z)/E(w)$

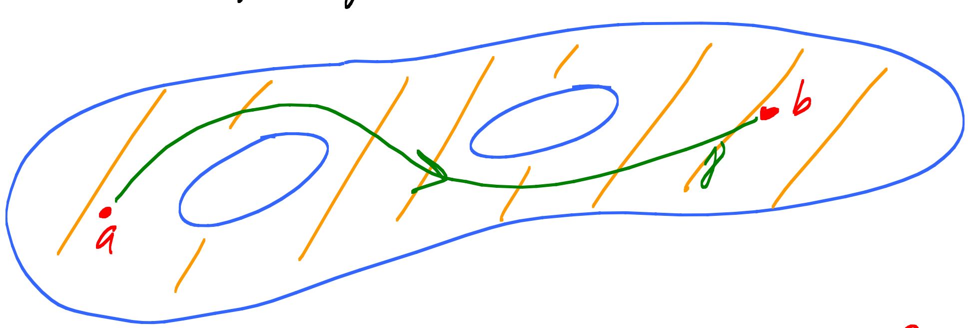
7) $E(x) = e^x$ when x is real.

8) $e^{i\theta} = \cos\theta + i\sin\theta$ and deMoivre's

9) $E(z) = 1 \iff z = 2\pi n i$, $n \in \mathbb{Z}$.

Lemma: If f is analytic on $D_r(a)$ and $f' \equiv 0$, then f is constant.

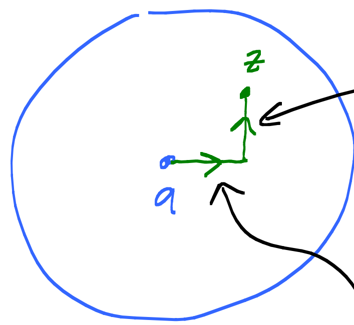
Pf: $f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \equiv 0 \implies \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$



$\int_\gamma \nabla u \cdot d\vec{s} = u(b) - u(a)$ ← Ouch! Don't know that u is C^1-smooth yet

Freshman calculus: Suppose h is continuous on $[a,b]$ and diff'ble on (a,b) and $h'(t) \equiv 0$ on (a,b). Then h is constant on $[a,b]$. Pf: MVT

Pf of lemma



$u_y, v_y \equiv 0$. So u, v const on vert.

$u_x, v_x \equiv 0$

So $u, v =$ const along horiz.

Lemma: If f analytic on $D_R(0)$ and

$$f'(z) = k f(z) \quad \text{there,}$$

Then $f(z) = C E(kz)$ where $C = f(0)$.

Pf: $\quad f' - kf = 0 \leftarrow$ Integrating factor $E(-kz)$.

$$f'(z) E(-kz) \underbrace{- k E(-kz)}_{} f(z) \equiv 0$$

$$\frac{d}{dz} E(-kz) = E'(-kz) \frac{d}{dz}[-kz] = -k E(-kz)$$

$$\frac{d}{dz}\left[f(z) E(-kz) \right] \equiv 0$$

Lemma $\Rightarrow$ $\boxed{f(z) E(-kz) \equiv C.}$

Plug in $z=0$. Get $C = f(0)$.

<u>Stop</u>. Start proof over using $f(z) = E(kz)$. $\leftarrow f(0)=1$

Satisfies $f'(z) = E'(kz)\cdot k = kE(kz) = kf(z)$ ✓

Box: $E(kz)E(-kz) = 1$

So $E(-kz) = \dfrac{1}{E(kz)}$ $\leftarrow$ $E(w)$ never zero!

Box: $f(z) = CE(kz)$. $C = f(0)$ ✓

<u>Pf of property 3</u>: $f(z) = E(z+w)$ (w fixed).

$$f'(z) = E'(z+w)\frac{d}{dz}[z+w] = E(z+w) = f(z) \quad \uparrow k=1$$

Lemma $\Rightarrow$ $f(z) = f(0)E(z)$

$E(z+w) = E(0+w)E(z)$ ✓

<u>Big fact</u>: f analytic $\Rightarrow$ f' ananalytic

$\Rightarrow$ f'' analytic $\Rightarrow$ ----

<u>and</u>: f' exists $\Rightarrow$ f cont.

f'' exists $\Rightarrow$ f' cont.

$\vdots$

} So all derivatives are continuous.

C-R Equs $\Rightarrow$ u, v are C^∞-smooth!

Assume big fact: See that u and v are harmonic.

Why: C-R Eqns $\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$

$$\begin{aligned} \tfrac{\partial}{\partial x}(A):& \quad u_{xx} = v_{xy} \\ \tfrac{\partial}{\partial y}(B):& \quad + \; u_{yy} = -v_{yx} \\ \hline & \quad \Delta u = 0 \end{aligned}$$

$v_{xy} = v_{yx}$ if v C^2-smooth.

Similarly for v.

Big question: Given real valued harmonic fcn u, can we find a v such that $u+iv$ is analytic?

Hmmm: Need a v with $\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$ $\nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$

Aha! $\nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} \leftarrow$ field conservative

Can find v if $\operatorname{Curl}\begin{pmatrix} -u_y \\ u_x \end{pmatrix} = 0$

$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -u_y & u_x & 0 \end{bmatrix} = \hat{k}\,\Delta u \equiv 0$$

Yes! Can find v on domains without holes.

Def^{n}: v is called a harmonic conjugate for u.

$\mathbb{R}^2$: $\nabla V = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

$$\frac{\partial}{\partial y}\left(V_x = F_1 \right)$$
$$\frac{\partial}{\partial x}\left(V_y = F_2 \right)$$

$$F_{1y} = V_{yx} = V_{xy} = F_{2x}$$

Must be equal

$$\text{Curl } \vec{F} = \left(F_{2x} - F_{1y} \right) \hat{k}$$