Lecture 5 Complex integration

Hwk 1 due Friday

Ratio test: Given $\sum_{n=1}^{\infty} z_n$. Suppose $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$ exists $= L$.

$L < 1$ Series converges $\leftarrow$ compare end of series with geom series

$L > 1$ Series diverges $\leftarrow$ terms $\not\to 0$

$L = 1$ test fails $\leftarrow$ $\sum \frac{1}{n^2}$, $\sum \frac{1}{n}$

Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $D_R(z_0)$ with R. of C. $= R$

$\lim_{n\to\infty} \sqrt[n]{n} = 1$. Hadamard's shows R. of. C. of

L'H

$\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = f'(z)$

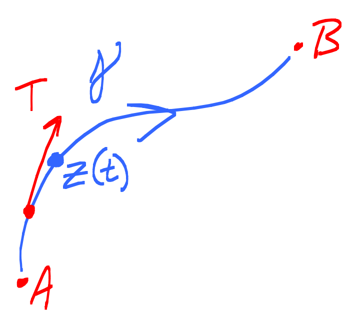
has same R. of C. R.

Big fact: Power series converge to analytic fcns in $D_R(z_0)$, $R =$ R. of C.

Bigger fact: Analytic fcns are given locally by conv. power series.

Stein: pp. 16-18. Pf's later.

Integration along paths:



$z(t) = x(t) + i y(t)$

$a \le t \le b$

$z : [a,b] \to \mathbb{C}$

Def^n: $z'(t_0) = \lim_{t\to t_0} \frac{z(t) - z(t_0)}{t - t_0} = \lim \left[DQ\, x(t) + i\, DQ\, y(t) \right]$

$= x'(t_0) + i y'(t_0)$

Tangent vector: $T = z'(t)$. Unit (complex) Tang. vect. $\hat{T} = \frac{z'(t)}{|z'(t)|}$

Def^{n}: "Trace of γ" $= tr(\gamma) = \{z(t) : a \le t \le b\}$.

Abuse of notation: $z \in \gamma$ short for $z \in tr(\gamma)$.

Chain rule #1: $g : \Omega_1 \to \Omega_2 \subset \mathbb{C}$ analytic

$f : \Omega_2 \to \mathbb{C}$ analytic

$h(z) = f(g(z))$ is analytic on Ω_1 and

$h'(z) = f'(g(z))\, g'(z)$ there. Pf. HWK.

Chain rule #2: f analytic and $z(t) = x(t) + i y(t)$ diff'ble,

Then $\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$

complex deriv.

Pf: $f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$

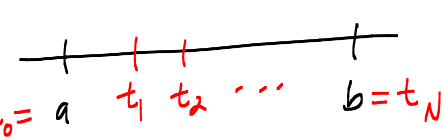
$$\frac{d}{dt} f(z(t)) = \Big(u_x \dot{x} + \overset{= -v_x}{u_y}\, \dot{y}\Big) + i\,\Big(v_x \dot{x} + \overset{= u_x}{v_y}\, \dot{y}\Big)$$

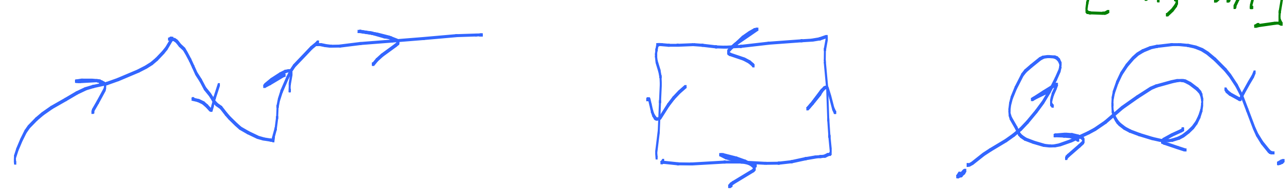
$$= \underbrace{\big[u_x + i v_x\big]}_{= f' \text{ red box } \#1} (\dot{x} + i \dot{y})$$

Piecewise $\mathcal{C}^1$-paths: $z(t)$ cont on $[a, b]$,

cont diff'ble on (t_n, t_{n+1}),

$z'(t)$ extends to be cont on $[t_n, t_{n+1}]$.





Def^n $\displaystyle\int_a^b z(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n)\Delta t_n \underset{\text{def}^n}{=} \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$

Fundamental Thm of Calculus #1: $\displaystyle\int_a^b z'(t)\,dt = z(b) - z(a)$

True for piecewise $\mathcal{C}'$ smooth $z(t)$ too.

$$\int_a^b z'(t)\,dt = \sum_{n=0}^{N-1} \underbrace{\int_{t_n}^{t_{n+1}} z'(t)\,dt}_{z(t_{n+1}) - z(t_n)} \underset{\text{telescopes}}{=} z(t_N) - z(t_0)$$

Lemma: $\displaystyle\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$

Think: $\displaystyle\left|\sum z(t_n)\Delta t_n\right| \le \sum |z(t_n)|\,\Delta t_n$ $\leftarrow$ let $\Delta t \to 0$

Pf: $\displaystyle\int_a^b z(t)\,dt = re^{i\theta}, \qquad \lambda = e^{-i\theta}$

$$\underbrace{\left|\int_a^b z(t)\,dt\right|}_{r,\ \text{real}} = \lambda\int_a^b z(t)\,dt \underset{\text{show this}}{=} \int_a^b \lambda z(t)\,dt$$

$$= \int_a^b \operatorname{Re} \lambda z(t)\,dt + i\underbrace{\int_a^b \operatorname{Im} \lambda z(t)\,dt}_{\text{must } = 0}$$

$$= \int_a^b \operatorname{Re} \lambda z(t)\,dt$$

$$\le \int_a^b |\operatorname{Re} \lambda z(t)|\,dt$$

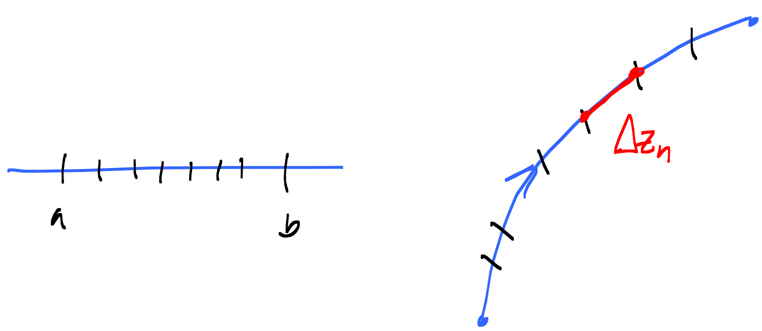
$$\boxed{|\operatorname{Re} w| \le |w|}$$

$$\le \int_a^b \underbrace{|\lambda z(t)|}_{\underbrace{|\lambda|}_{=1}\,|z(t)|}\, dt$$

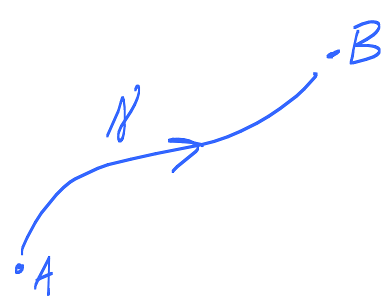
<u>Def^n</u>: Suppose $f: tr(\gamma) \to \mathbb{C}$ is continuous. Then

$$\int_\gamma f\, dz \overset{\text{Def}^n}{=} \int_a^b f(z(t))\, \underbrace{z'(t)\, dt}_{dz}$$

Konrad Knopf: $\displaystyle\int_\gamma f\, dz = \lim_{\Delta t \to 0} \sum f(z(t_n))\, \underbrace{\Delta z_n}_{\underbrace{z(t_{n+1}) - z(t_n)}_{\approx\, z'(t_n)\, \Delta t_n}}$



<u>Fund Thm Calculus #2</u>:



f analytic on an open set containing $tr(\gamma)$ and

f' continuous on the open set, then $\displaystyle\int_\gamma f'\, dz = f(B) - f(A)$

<u>Note</u>: Later we will show f analytic $\Rightarrow f'$ analytic $\Rightarrow f''$ analytic

f' continuous

Don't need red stuff.

<u>Pf:</u> $\int_\gamma f' \, dz = \int_a^b f'(z(t)) \; z'(t) \, dt = \int_a^b \frac{d}{dt}\left[f(z(t))\right] dt$

Chain rule #2

$= f(z(b)) - f(z(a))$

Fund Thm Calc #1

$= f(B) - f(A)$ ✓

<u>Lemma:</u> (Basic Estimate):

$$\left| \int_\gamma f \, dz \right| \leq \left(\max_{z \in tr(\gamma)} |f(z)| \right) \text{Length}(\gamma)$$

where $\text{Length}(\gamma) = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} \, dt = \int_a^b |z'(t)| \, dt$

<u>Pf:</u> $\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(z(t)) \; z'(t) \, dt \right|$

$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq \text{Max}} \; |z'(t)| \, dt \leq M \int_a^b |z'(t)| \, dt$ ✓

$\leq$ Max (Cont fcn on closed $[a,b]$)

$= M$

<u>Prob:</u> $\sum a_n z^n$, $a_n \in D_1(0)$ rational real and imag.

$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq 1$ So $R \geq 1$

easy.

Plug $z = 1$, terms $\not\to 0$. Diverges. So $R \leq 1$ too.