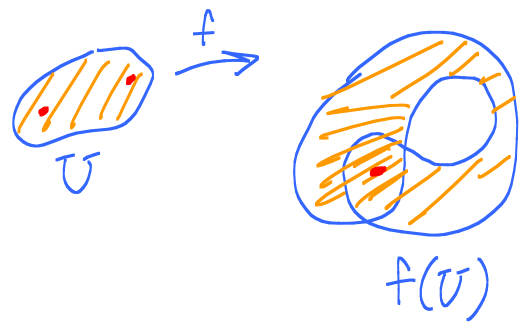
Lecture 16 The Open Mapping Theorem (OMT)

HWK 5 on home page due Fri
Midterm in class Mon, March 4

OMT: Non-constant analytic fcns map open sets to open sets

Notation: $f(U) = \{f(z) : z \in U\}$



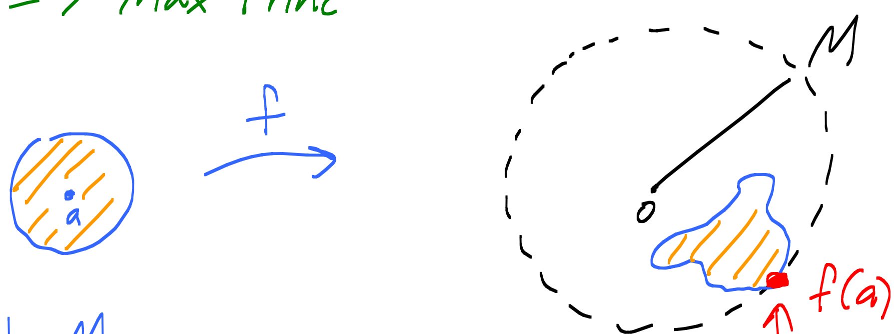
$f : \Omega \to \mathbb{C}$. $f^{-1}(U) = \{z \in \Omega : f(z) \in U\}$

Fact: f continuous $\iff$ $f^{-1}(U)$ is open whenever U is.

Cor of OMT: If analytic f has an inverse, then the inverse is continuous. [Later: this will used to see that the inverse is analytic.]

OMT Way false $\mathbb{R} \to \mathbb{R}$: $h(t) = 1 - t^2$ $\quad h((-1,1)) = (0,1]$

Remark: OMT $\Rightarrow$ Max Princ



$|f(a)| = M$
max on $\overline{D_r(a)}$

oops!
No way to put disc in $f(D_r(a))$ at $f(a)$.

Argument Principle for a disc: Suppose f is analytic on $D_R(z_0)$ and $0 < r < R$ and suppose f has no zeroes on the circle $C_r(z_0)$. Then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with mult.}\end{array}\right)$$

Pf: Suppose $f(a)=0$. Then $f(z)=(z-a)^N F(z)$

where F is analytic and $F(a)\neq 0$.

$$\frac{f'(z) = N(z-a)^{N-1}F(z) + (z-a)^N F'(z)}{f(z) = (z-a)^N F(z)}$$

$$\frac{f'}{f} = \underbrace{\frac{N}{z-a}}_{\text{Princ. part}} + \frac{F'(z)}{F(z)}$$

$N \leftarrow \operatorname{Res}_a f$

f has a simple pole at $z=a$.

$\left(N = \text{Residue of } f \text{ at } a \right)$

Let $a_1, \dots, a_N$ be zeroes inside $C_r(z_0)$ of mult $m_1, \dots, m_N$ respectively.

$$\frac{f'}{f} - \sum_{n=1}^{N} \frac{m_n}{(z-a_n)} = h$$

$\leftarrow$ has removable sing at each a_n. Give it proper values.

$$\int_{C_r(z_0)} \frac{f'}{f}\,dz - \sum_{n=1}^{N} m_n \underbrace{\int_{C_r(z_0)} \frac{1}{z-a_n}\,dz}_{2\pi i} = \int_{C_r(z_0)} h\,dz = 0$$

$\uparrow$ by Cauchy's Thm.

Done! $\sum_{n=1}^{N} m_n \leftarrow$ counting zeroes with mult.

Argument Princ #2: If f is allowed to have poles inside $C_r(z_0)$:

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes } f \\ \text{counted with mult} \\ \text{inside } C_r(z_0)\end{array}\right) - \left(\begin{array}{c}\#\text{ poles } f \\ \text{counted with mult} \\ \text{inside } C_r(z_0)\end{array}\right)$$

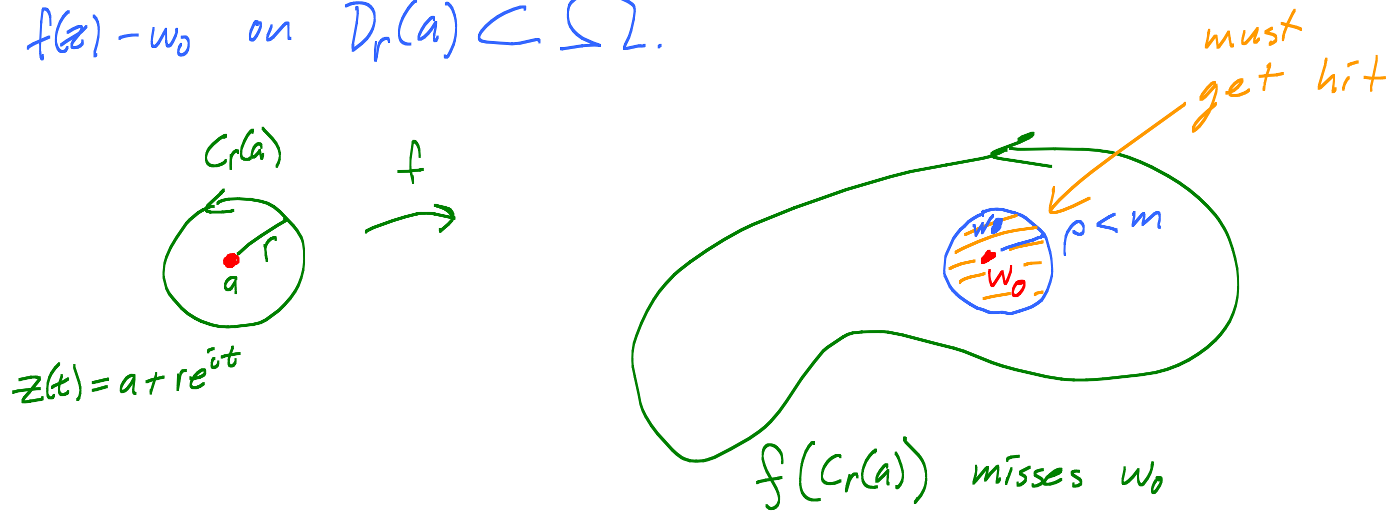
Pf: Pole of order N: $\quad f(z) = (z-a)^{-N} F(z)$

$$\frac{f'}{f} = \frac{-N}{z-a} + \frac{F'}{F}$$

Pf of OMT: f analytic on a domain Ω and non-constant,

Given $w_0 \in f(\Omega)$. Then $\exists\, a \in \Omega$ such that $f(a) = w_0$.

f not constant, So $f(z) - w_0$ has an isolated zero at a. So $\exists\, r > 0$ such a is the only zero of $f(z) - w_0$ on $\overline{D_r(a)} \subset \Omega$.



$|f(a+re^{it}) - w_0| \leftarrow$ has a positive min m on $[0, 2\pi]$

Pick ρ with $0 < \rho < m$

Want to show every $w \in D_\rho(w_0)$ gets hit by f on $D_r(a)$, i.e.,

$f(z) - w$ has a zero inside $C_r(a)$.

Hmmm: $\left(\begin{array}{c} \#\text{ zeroes of} \\ f(z) - w \\ \text{inside } C_r(a) \end{array} \right) = \dfrac{1}{2\pi i} \displaystyle\int_{C_r(a)} \frac{f'(z)}{f(z) - w}\, dz \leftarrow$ want this ≥ 1

Aha! $H(w) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f'(z)}{f(z)-w}\, dz$ is <u>analytic</u> in w!

And integer valued. And $H(w_0)$ = mult of zero of $f(z)-w_0$ at a ≥ 1.

Done, if I show H analytic.

<u>Fact</u>: A continuous integer valued fcn on a domain must be constant.

$$H(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(z(t))}{f(z(t))-w}\, z'(t)\,dt$$

$C_r(a) : z(t)$

$f(C_r(a)) : f(z(t))$

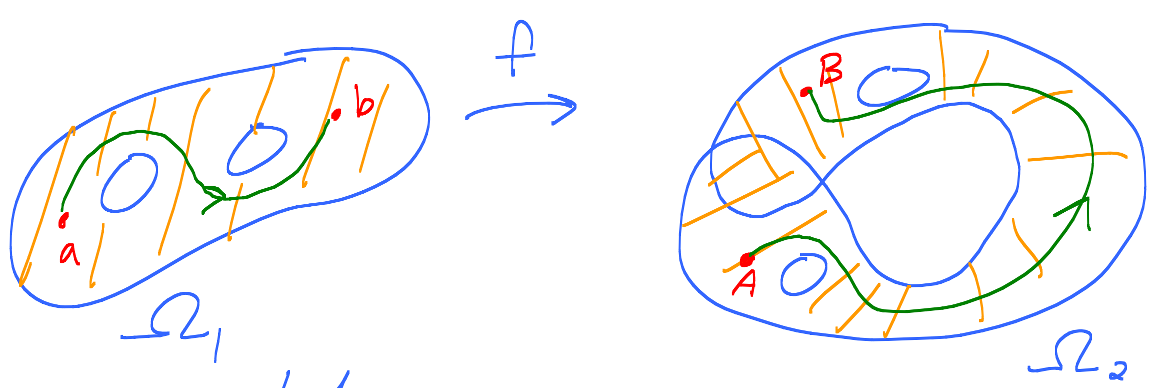
$$= \frac{1}{2\pi i}\int_{f(C_r(a))} \frac{1}{\zeta - w}\, d\zeta$$

showed analytic in Prob 1, HWK 2.

Conclude $D_\rho(w_0) \subset f(D_r(a)) \subset f(\Omega)$ ← open ✓

<u>Cor</u>: Non-constant analytic f map <u>domains</u> to <u>domains</u>.

<u>Why</u>:



OMT $\Rightarrow$ Ω_2 open

See Ω_2 connected ✓

Next time: OMT $\Rightarrow$ Strong inverse fcn thm for analytic fcns

HWK prob

$F(a) = 0$

$g(a) = 0$

Given F analytic : $\Omega \to D_1(0)$ one-to-one and onto.

Then $|g'(a)| \le |F'(a)|$.

F^{-1} is analytic $\leftarrow$ strong I.F.T.

Schwarz: $h = g \circ F^{-1} : D_1(0) \to D_1(0)$, $h(0) = 0$.