Lecture 23 Linear Fractional Transformations (LFTs) $L(z)=\frac{az+b}{cz+d}$

$(ad-bc)\neq 0$

Fact: $L \in LFT$. Then $L^{-1} \in LFT$.

$$w=\frac{az+b}{cz+d} \quad \xleftarrow{\text{solve for } z} \quad z=\frac{dw-b}{-cw+a}$$

Fact: LFTs form a group. $\left(Id = \frac{1\cdot z + 0}{0\cdot z+1} = z \right)$

Fact: LFTs: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ 1-1 onto

Why: Case $c \neq 0$. $L(z)=\frac{az+b}{cz+d} \xrightarrow[z\to -\frac{d}{c}]{} \infty$

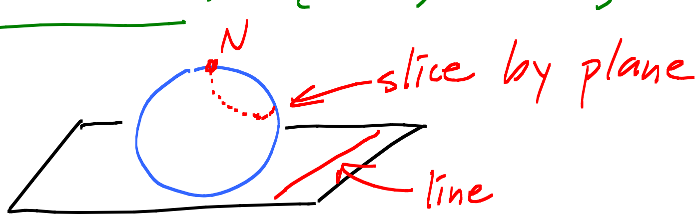
Define $L\left(-\frac{d}{c}\right)=\infty$

Also, $\lim_{z\to\infty} L(z) = \frac{a}{c}$. Define $L(\infty)=\frac{a}{c}$

Case $c=0$: $L(z)=Az+B$, $A\neq 0$

$L(\infty)=\infty$

Fun fact: $\{$lines, circles$\}$ in $\mathbb{C}$ $\sim$ $\{$circles$\}$ in $\hat{\mathbb{C}}$



Fact: LFTs $\underbrace{\frac{az+b}{cz+d}}_{L} \sim \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbb{A}}$

$$L^{-1} \sim \mathbb{A}^{-1} = \frac{1}{\text{Det}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

cancels out in L

$$L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2$$

$$z \mapsto z \sim \mathbb{I}$$

Lemma: An LFT that fixes 3 distinct points must be the identity.

Pf: $L(z_j) = z_j \qquad j = 1, 2, 3$

$$\frac{az_j + b}{cz_j + d} = z_j$$

$$\boxed{c z_j^2 + (d-a) z_j - b = 0}$$

Quadratic poly with 3 roots must be $\equiv 0$!

So $c = 0$, $a = d$, $b = 0$. $\quad L(z) = \frac{az + 0}{0 \cdot z + d} = \frac{a}{d} z = z$

so $\det = ad \neq 0$. So $a \neq 0$, $d \neq 0$.

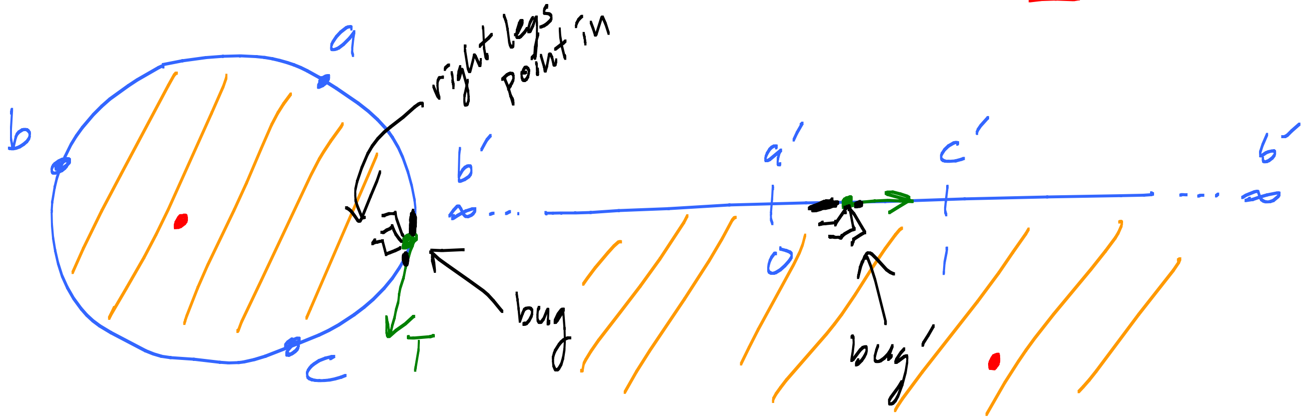
Cor: If two LFTs agree at 3 distinct points, they are the same.

Note: 3 points determine a circle.
2 points determine a line (plus pt at ∞)

Useful LFTs: EX: $\dfrac{z-a}{z-b}$ $\quad a \mapsto 0$, $\; b \mapsto \infty$

Aha! $L(z) = \dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}$ $\quad a \mapsto 0$, $\; b \mapsto \infty$, $\; c \mapsto 1$

EX:

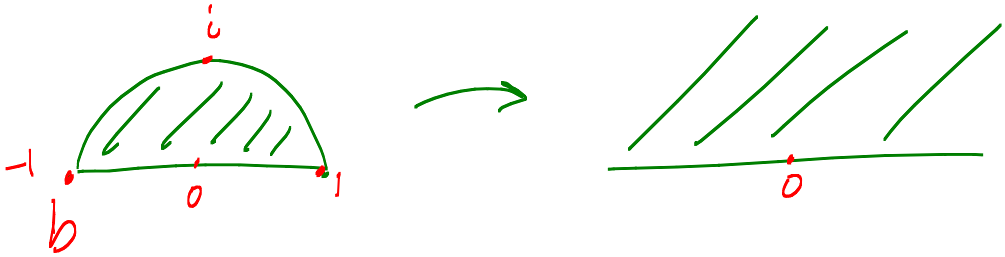


Or check which half plane one pt goes to.

Prob: Find an LFT: $z_1 \mapsto w_1$, $\; z_2 \mapsto w_2$, $\; z_3 \mapsto w_3$

L_1: $z_1 \mapsto 0$, $\; z_2 \mapsto \infty$, $\; z_3 \mapsto 1$ $\qquad L_2$: $w_1 \mapsto 0$, $\; w_2 \mapsto \infty$, $\; w_3 \mapsto 1$

Ans: $L = L_2^{-1} \circ L_1$



$L(b) = \infty \longrightarrow$ quadrant $\longrightarrow$ first quad $\xrightarrow{z^2}$ UHP