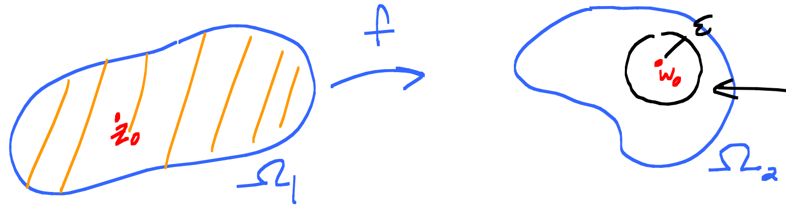
Lecture 31 Solution of the Dirichlet problem, Schwarz's Theorem

Fact: $f : \Omega_1 \to \Omega_2$ analytic, u harmonic on Ω_2. Then $u \circ f$ harmonic on domain Ω_1.

$\overline{D_\varepsilon(w_0)} \subset \Omega_2$

Why:



Get $\overbrace{u + iv}^{g}$ analytic on $D_\varepsilon(w_0)$.

f cont. So $\exists\, \delta > 0$ such that $f(D_\delta(z_0)) \subset D_\varepsilon(w_0)$.

Finally $\quad u \circ f = \mathrm{Re}\,[g \circ f] \leftarrow$ analytic on $D_\delta(z_0)$.

$\mathrm{Re}\,[\text{analytic}] = \text{harmonic}$ ✓

How to cook up Poisson Kernel: Ave prop:

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta$$

Hmmm: $u(\varphi_{-a}(z))$ is harmonic on $D_1(0)$ where $\varphi_{-a}(z) = \dfrac{z + a}{1 + \bar{a} z}$

$\varphi_a(a) = 0$

$$u(a) = u(\varphi_{-a}(0)) = \frac{1}{2\pi} \int_0^{2\pi} u\left(\underbrace{\frac{e^{i\theta} + a}{1 + \bar{a} e^{i\theta}}}_{e^{i\psi}} \right) d\theta$$

Hairy calculus exercise: $\quad d\theta = \dfrac{1 - |a|^2}{|e^{i\psi} - a|^2}\, d\psi$

So $\quad u(a) = \displaystyle\int_0^{2\pi} u(e^{i\psi})\, P(a, \psi)\, d\psi$

$\uparrow$ or $e^{i\psi}$

Stein: p. 66: 11.

Fact: u continuous on $\overline{D_1(0)}$ and harmonic inside.

Then $u =$ Poisson integral.

Why: Easy if u harmonic on $D_R(0)$ where $R>1$.

Trick: Given u cont $\overline{D_1(0)}$, harmonic inside.

$u(rz)$ where $0<r<1$ is harmonic on bigger disc.

$u(rz) =$ Poisson integral. Let $r \nearrow 1$.

Get $u = P.$ int. ✓

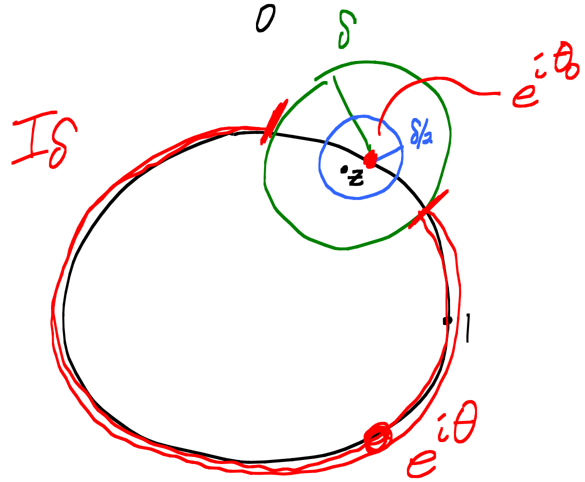
Schwarz's Theorem: Poisson integral formula solves the D. Prob.

Pf: 3 properties

1) $P(a,\theta) > 0$ ✓ from formula

2) $\displaystyle\int_0^{2\pi} P(a,\theta)\, d\theta = 1$ ✓ $u(z) \equiv 1$ is harmonic!

3)



$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$$P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$$

$\longrightarrow 0$ uniformly on I_δ

for $z \in D_{\delta/2}(e^{i\theta_0})$

Easy! $\frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{i\theta}|^2} \le \frac{1}{2\pi}\frac{1-|z|^2}{\left(\frac{\delta}{2}\right)^2} = \frac{2}{\pi\delta^2}(1-|z|)\underbrace{(1+|z|)}_{<1+1=2}$ ✓

4) $P(z,\theta)$ is harmonic in z

because $P(z,\theta) = \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$

<u>Pf</u>: <u>Step 1</u>: Define

$$u(z) = \int_0^{2\pi} P(z,\theta)\,\varphi(\theta)\,d\theta$$

u is harmonic: $u(z) = \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,\varphi(\theta)\,d\theta\right]$

$$= \mathrm{Re}\left[\frac{1}{2\pi}\,z\underbrace{\int_0^{2\pi}\frac{\varphi(\theta)\quad ie^{i\theta}\,d\theta}{(e^{i\theta}-z)\;ie^{i\theta}}}_{\int_{C_1(0)}\frac{[\varphi(w)/iw]}{w-z}\,dw} + \frac{1}{2\pi}\int_{C_1(0)}\frac{1}{i}\,\frac{\varphi(w)}{w-z}\,dw\right]$$

Analytic! HWK 2: 1.

<u>Step 2</u>: Show $\lim_{z\to e^{i\theta_0}} u(z) = \varphi(\theta_0)$.

$u(z) - \varphi(\theta_0) =$

$$\underbrace{\int_0^{2\pi} P(z,\theta)\left[\varphi(\theta)-\varphi(\theta_0)\right]}_{I} d\theta$$

because $\int P\,d\theta = 1$

Let $\varepsilon > 0$. $\exists\ \delta > 0$ such that $|\varphi(\theta)-\varphi(\theta_0)| < \varepsilon$ when $e^{i\theta} \in D_\delta(e^{i\theta_0})$.

$$\text{Now} \quad I = \int_{I_\delta} \underbrace{P(z,\theta)}_{\text{small}} \big(\varphi(\theta)-\varphi(\theta_0)\big)\,d\theta + \int_{[0,2\pi]-I_\delta} P(z,\theta) \underbrace{\big(\varphi(\theta)-\varphi(\theta_0)\big)}_{\text{small}}\,d\theta$$

$$|I| \le \underbrace{\int_{I_\delta} P(z,\theta)\,|\varphi(\theta)-\varphi(\theta_0)|\,d\theta}_{I_1} + \underbrace{\int_{[0,2\pi]-I_\delta} P(z,\theta)\,|\varphi(\theta)-\varphi(\theta_0)|\,d\theta}_{I_2}$$

$$I_1 \le \frac{4}{\pi\delta^2}\big(1-|z|\big)\,2M \cdot \underbrace{\text{Length}(I_\delta)}_{< 2\pi}$$

$M = \text{Max}\,|\varphi|$

when $z \in D_{\delta/2}(e^{i\theta_0})$.

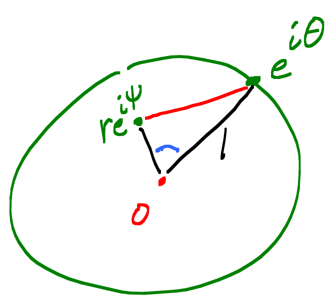
$$I_2 \le \int_{[0,2\pi]-I_\delta} P(z,\theta)\cdot\varepsilon\,d\theta$$

$$< \varepsilon \underbrace{\int_{[0,2\pi]} P(z,\theta)\,d\theta}_{=1}$$

because $P > 0$

Done ✓

Other formulas :



blue angle $= \psi - \theta$

red line $= \sqrt{1 + r^2 - 2r\cos(\psi-\theta)}$

$$u(re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\psi-\theta)}\,\varphi(\theta)\,d\theta$$

$D_R(z_0)$ version:

$$u(z_0+re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2rR\cos(\psi-\theta)+r^2}\,u(z_0+Re^{i\theta})\,d\theta$$

Note: $r=0$ gives ave property.

Explosion of theorems!

EX: Uniform limits of harmonic fcns are harmonic:

$$u_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\,u_n(e^{i\theta})\,d\theta$$

$$\downarrow \qquad\qquad \downarrow$$

$u(z) = \frac{1}{2\pi}$ Poisson integral of $u(e^{i\theta})$ ✓ harmonic!