Lecture 33: The weak averaging property

Weak averaging property (WAP): A continuous fcn on a domain Ω satisfies the W.A.P. if, given any pt $z_0 \in \Omega$, $\exists\ \varepsilon > 0$ such that

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \quad \text{for } r \text{ with } 0 < r < \varepsilon.$$

Thm: WAP $\Longrightarrow$ harmonic.

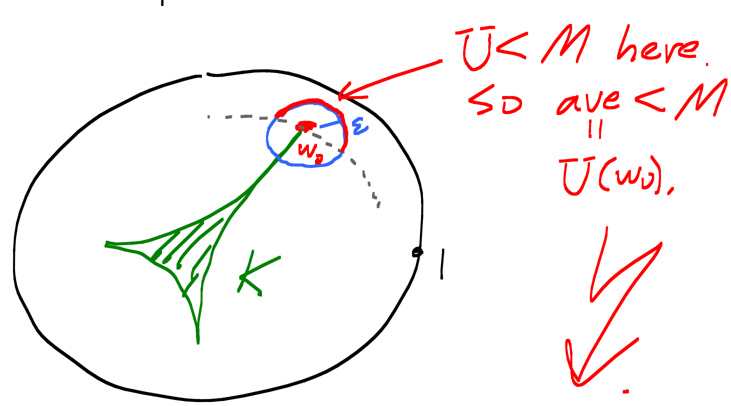
Pf: Can suppose u continuous on $\overline{D_1(0)}$ and satisfies the WAP on $D_1(0)$. Let $\tilde{u}$ be the harmonic extension to $D_1(0)$ of $u(e^{i\theta})$ on $C_1(0)$. Let $U = \tilde{u} - u$. Note that $U \equiv 0$ on $C_1(0)$. Suppose $\exists\ z_0 \in D_1(0)$ such that $U(z_0) > 0$. Let $M = \max\limits_{\overline{D_1(0)}} U$. Then $M > 0$.

Define $K = \{ z \in \overline{D_1(0)} : U(z) = M \}$.

Note K is compact and $K \cap C_1(0) = \emptyset$.

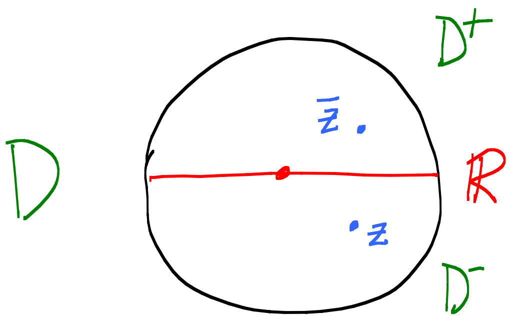
So $K \subset\subset D_1(0)$.

Claim: $\exists\ w_0 \in K$ with maximum modulus. (because $|z|$ is a cont. fcn on a compact set, so assumes max.)



So no such z_0 exists. Similarly, no such place where $U(z_0) < 0$. [Replace $\tilde{u} - u$ by $u - \tilde{u}$ and repeat.]

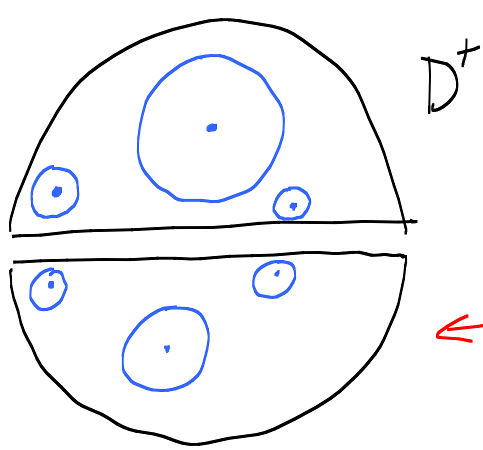
Reflection principle for harmonic fcns: Suppose u is real valued and continuous on $D \cap \{\operatorname{Im} z \geq 0\}$ and harmonic on D^+ and $u(x) = 0$ for $x \in \mathbb{R} \cap D$.



Define $$U(z) = \begin{cases} u(z), & z \in D \cap \{\operatorname{Im} z \geq 0\} \\ -u(\bar{z}), & z \in D^- \end{cases}$$

Then U is harmonic on D.

Pf:

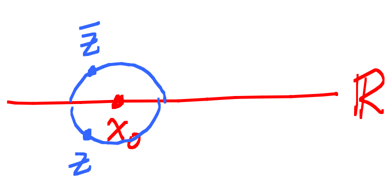


WAP!

$\underset{u(x,y)}{u}$ harmonic $\Rightarrow$ $\underbrace{u(\bar{z})}_{u(x,-y)}$ harmonic

So WAP on D^-!

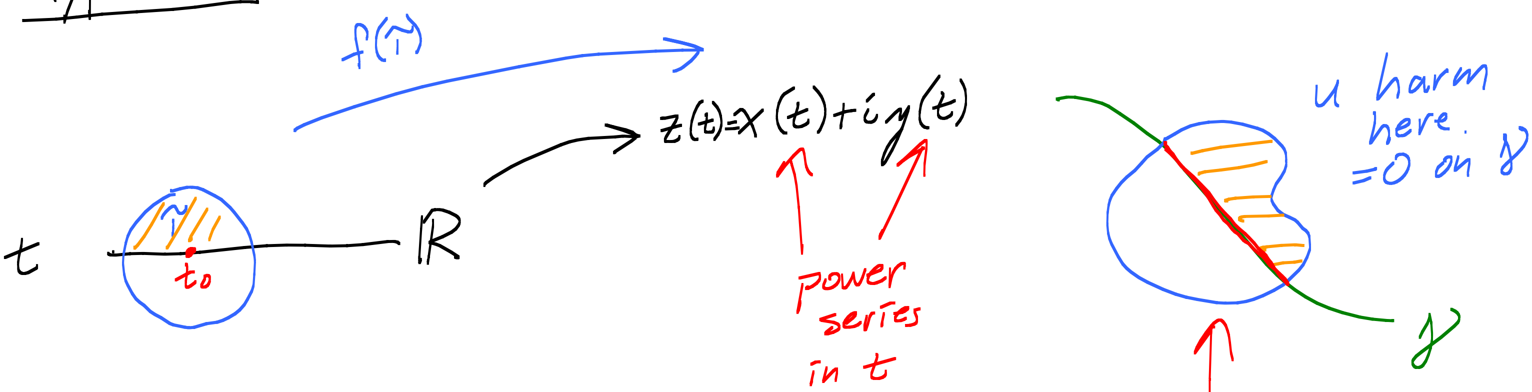
Last step



$$0 = u(x_0) = \underbrace{\int_{Top} + \int_{Bot}}_{\text{cancel because}}$$

$u(\bar{z}) = -u(z)$

<u>Typical use</u>: "Real analytic curves"



Can reflect to other side of γ as harm fcn.

$z'(t_0) \Rightarrow f'(t_0) \neq 0$ too

<u>Schwarz reflection fcn</u>: $f\left(\overline{f^{-1}(w)}\right) = R(w)$

antiholomorphic $\leftarrow$ conjugate is analytic.

<u>Facts</u>: 1) $R(R(z)) = z$

2) $R(z) = z$ on γ

3) R is antiholomorphic.

<u>EX</u>: $\gamma = C_1(0)$. Schwarz fcn: $S(z) = \frac{1}{z}$

"meromorphic" on $D_1(0)$

$S(z) = \bar{z}$ on $C_1(0)$.

Schwarz reflection fcn $R(z) = 1/\bar{z}$

"Something about Poisson and Dirichlet" www.math.purdue.edu/~bell/papers at top.

Easy to solve D. Prob given poly data:

<u>Step 1</u>: $p(x,y) = $ real poly $= \sum_{0 \le n,m \le N} c_{nm} x^n y^m$

$$p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \sum_{0}^{N} A_{nm} z^n \bar{z}^m$$

<u>Cases</u>: <u>$n=m$</u>: $z^n \bar{z}^n = |z^2|^n = 1$ on $C_1(0)$

harmonic extension!

<u>$n<m$</u>: $z^n \bar{z}^m = \underbrace{z^n \bar{z}^n}_{=1 \text{ on } C_1(0)} \bar{z}^{m-n} = \overline{z^{m-n}}$ on $C_1(0)$

harmonic!

<u>$n>m$</u>: $z^n \bar{z}^m = z^{m-n} \underbrace{z^m \bar{z}^m}_{=1 \text{ on } C_1(0)} = z^{m-n}$ on $C_1(0)$

harmonic!

<u>Aha!</u> Harmonic extension of $p(x,y)$ is

$$c_0 + \overline{P(z)} + Q(z) \quad \text{is a poly.}$$

P, Q polys

<u>Khavinson-Schapiro conjecture</u>: The disc and the ellipse are the only domains with poly data $\longrightarrow$ poly sol^n to D. Prob property.

<u>Why the ellipse</u>: $E = \left\{ \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}_{r(x,y) \,=\, \text{poly defining fcn.}} < 0 \right\}$

$\tilde{\Delta}$: Fischer map: $\Delta(r\cdot p)$: $\begin{matrix}\text{polys} \\ \text{deg} \\ N \text{ or less}\end{matrix} \xrightarrow{\text{linear}} \underbrace{\begin{matrix}\text{polys} \\ \text{deg} \\ N \text{ or less}\end{matrix}}_{\mathcal{P}_N}$

<u>Claim</u>: $\tilde{\Delta}$ is 1-1: $\Delta(\underbrace{r\cdot p}_{\text{harmonic!}}) \equiv 0$

But $r \equiv 0$ on bndry.

Max Princ $\Longrightarrow rp \equiv 0$.

$\Longrightarrow p \equiv 0$.

$\tilde{\Delta} : \mathcal{P}_N \longrightarrow \mathcal{P}_N$ linear, 1-1,

$\Longrightarrow$ <u>onto</u>!

Use it to solve D. Prob.

Given poly data p, get q with $\Delta(r\cdot q) = \Delta p$

Then $\underbrace{p - r\cdot q}_{= p \text{ on bndry}}$ solves D. Prob with data p.

$\Delta(p - r\cdot q) = \Delta p - \Delta p \equiv 0$. harmonic ✓.