Lecture 39 Inhomogeneous Cauchy-Riemann equations

Review problems for final exam on home page.

The $\bar\partial$-operator:

$$\begin{cases} dz = dx + i\,dy \\ d\bar z = dx - i\,dy \end{cases}$$

$$\begin{cases} dx = \dfrac{dz + d\bar z}{2} \\ dy = \dfrac{dz - d\bar z}{2i} \end{cases}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \Bigg(\underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial z}}\Bigg)dz + \Bigg(\underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial \bar z}}\Bigg)d\bar z$$

Definitions

Fact: $f = u + iv$. u & v satisfy the C-R Eqns $\Longleftrightarrow \dfrac{\partial f}{\partial \bar z} = 0$.

And, if f is analytic, then $\dfrac{\partial f}{\partial z} = f'$

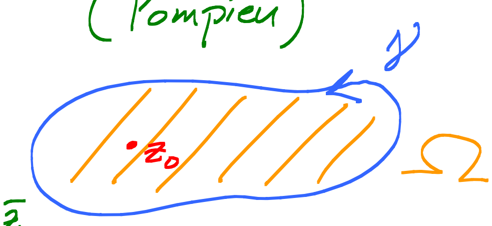
Think: $f = \sum a_{nm} x^n y^m = \sum \alpha_{nm} z^n \bar z^m$

Think: f analytic means no $\bar z$ terms.

Complex chain rule: $w = f(\zeta)$ where $\zeta = g(z)$

$$\begin{cases} \dfrac{\partial w}{\partial z} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial z} + \dfrac{\partial w}{\partial \bar\zeta}\dfrac{\partial \bar\zeta}{\partial z} \\[2ex] \dfrac{\partial w}{\partial \bar z} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial \bar z} + \dfrac{\partial w}{\partial \bar\zeta}\dfrac{\partial \bar\zeta}{\partial \bar z} \end{cases}$$

Improved Cauchy Integral Formula (Pompieu)



$$u(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{u(z)}{z - z_0}\,dz + \frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial u}{\partial \bar z}(z)}{z - z_0}\,dz \wedge d\bar z$$

Note: u analytic $\Rightarrow \dfrac{\partial u}{\partial \bar z} \equiv 0$. Standard Cauchy int. formula.

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy)$$

$$= \underbrace{dx \wedge dx}_{=0} + i\,\underbrace{dy \wedge dx}_{-dx \wedge dy} - i\,dx \wedge dy + \underbrace{dy \wedge dy}_{=0}$$

$$= -2i\,dx \wedge dy \leftarrow \text{area measure}$$

Theorem: (Lars Hörmander, Several complex variables, Chapt. 1)
Given a complex valued C^∞ fcn v on a domain Ω, there is a $u \in C^\infty(\Omega)$ with $\underbrace{\frac{\partial u}{\partial \bar{z}} = v}_{\text{Inhomog C-R Eqns.}}$.

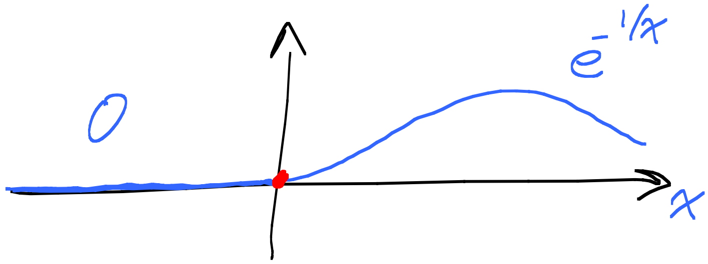
Note: If u_1 and u_2 both solve problem, then

$$\frac{\partial}{\partial \bar{z}}(u_1 - u_2) = v - v \equiv 0. \quad \text{So } u_1 - u_2 = \text{analytic}.$$

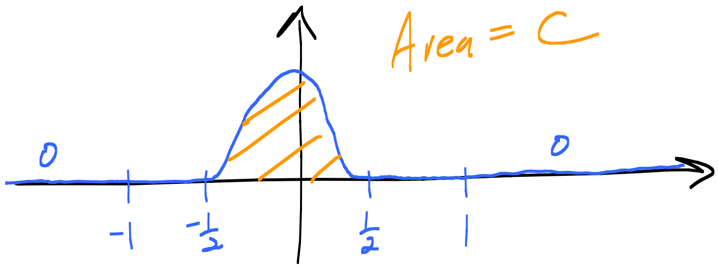
Cor: Mittag-Leffler Thm holds on any domain. Given a discrete $A \subset \Omega$ (write $A = \{a_n\}_{n=1}^\infty$, no limit pts in Ω) and principle parts $R_n(z)$, $\exists$ mero fcn f on Ω with p.p. R_n at a_n. [$f - R_n$ has a removable sing at a_n and f is analytic on $\Omega - A$.]

Pf: Plan. Step 1: Cook up a C^∞ smooth sol^n.

Step 0: Cook up C_0^∞ bump fcns and cut-off fcns.

$h(x)$ 

is C^∞-smooth

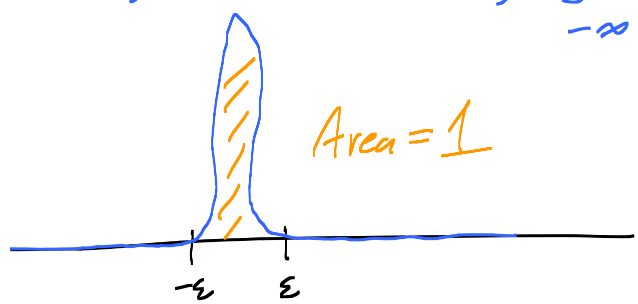
$\Psi(x)$ 

$h(x+\tfrac{1}{2})\,h(-x+\tfrac{1}{2})$

$\varphi(x) = \frac{1}{C}\Psi(x)$ even, $C_0^\infty(-\tfrac{1}{2},\tfrac{1}{2})$, $\int_{-\infty}^{\infty}\varphi\,dx = 1$,

$\varphi \geq 0$.

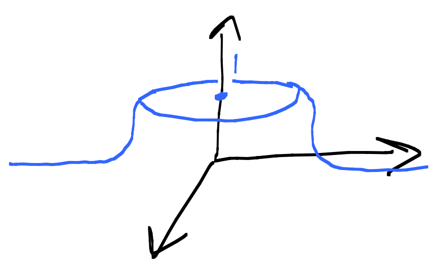
$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$

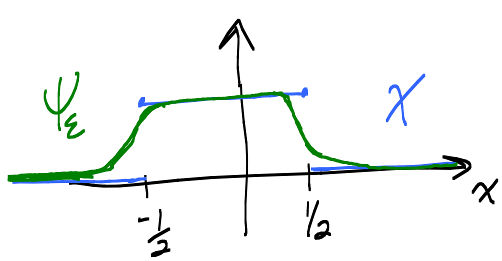


Classic fact: $\varphi_\varepsilon * f$ ← smooth and "close" to f.

↑ not smooth

C^∞ "birthday cake" fcn.





$$\underbrace{(\varphi_\varepsilon * \chi)(x)}_{\Psi_\varepsilon} = \int_{-\infty}^{\infty}\varphi_\varepsilon(x-y)\,\chi(y)\,dy$$

Birthday cake fcn $\quad \chi_r^{z_0}(z) = \Psi\left(\frac{|z-z_0|^2}{r^2}\right)$

centered at z_0 supported in $D_r(z_0)$ and C^∞-smooth and $\equiv 1$ on a nbhd of z_0.

Pf of M-L Thm: $\{a_n\}$ discrete. So $\exists$ discs $\overline{D_{r_n}(a_n)} \subset \Omega$ and $\overline{D_{r_n}(a_n)} \cap \overline{D_{r_m}(a_m)} = \emptyset$ for $n \neq m$. Let $$U = \sum_{n=1}^{\infty} \chi_n R_n$$

where χ_n is a C^∞ cut-off supported in $D_{r_n}(a_n)$ and $\equiv 1$ on a nbhd of a_n.

U is a "C^∞ solution" to the M-L problem.

Hmmm $$\frac{\partial U}{\partial \bar{z}} = \sum_{n=1}^{\infty} \left(\frac{\partial \chi_n}{\partial \bar{z}} \cdot R_n + \chi_n \frac{\partial R_n}{\partial \bar{z}} \right)$$

zero near a_n

bad at a_n

zero. but not defined at a_n

Aha! Define $$v = \begin{cases} 0 & \text{at } a_n\text{'s} \\ \dfrac{\partial U}{\partial \bar{z}} & \text{on } \Omega - A \end{cases}$$

v is C^∞-smooth on Ω.

Get $u \in C^\infty(\Omega)$ that solves $\frac{\partial u}{\partial \bar z} = V$.

<u>Claim</u>: $f = U - u$ solves M-L Prob!

Near a_n: $\underbrace{U - u}_{f} = \chi_n R_n - u$

$\uparrow$ C^∞-smoth

$f - R_n = \underbrace{(\chi_n - 1)}_{\substack{\text{zero} \\ \text{near } a_n}} R_n - u \leftarrow$ bndd

See a_n removable sing.

$\frac{\partial f}{\partial \bar z} = \frac{\partial U}{\partial \bar z} - V \equiv 0$ on $\Omega - A$. So f analytic on $\Omega - A$.

<u>Fact</u>:
$$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{v(z)}{z - z_0}\, dz \wedge d\bar z$$

solves $\frac{\partial u}{\partial \bar z} = v$ when $v \in C_0^\infty(\Omega)$.

<u>Radó's Theorem</u>: If f is a continuous complex valued fcn on a domain Ω that is analytic where it is <u>not</u> zero, then it is analytic.