Lecture 3 C-R eqns and the complex exponential

Last time: u C^1-smooth:

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{\frac{\partial u}{\partial x}}_{u_x^0}(x_0,y_0)(x-x_0) + \underbrace{\frac{\partial u}{\partial y}}_{u_y^0}(x_0,y_0)(y-y_0) + R_u(x,y)$$

where $\dfrac{R_u(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \longrightarrow 0$ as $(x,y) \to (x_0,y_0)$

$z = x+iy$ $\qquad$ $z_0 = x_0 + iy_0$

$\sqrt{} = |z-z_0|$

Theorem: u,v C^1-smooth and satisfy the C-R eqns.

Then $f(x+iy) = u(x,y) + iv(x,y)$ is analytic.

Pf: $DQ = \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{(u+iv)-(u^0+iv^0)}{z-z_0}$

$$= \frac{\left[u_x^0\cdot(x-x_0) + \overset{-v_x^0}{\overset{\shortparallel}{u_y^0}}\cdot(y-y_0)\right] + i\left[v_x^0\cdot(x-x_0) + \overset{u_x^0}{\overset{\shortparallel}{v_y^0}}\cdot(y-y_0)\right]}{z-z_0} + \underbrace{\frac{R_u + iR_v}{z-z_0}}_{\mathcal{E}}$$

Aha!

$$= \frac{[u_x^0 + iv_x^0]\cdot\cancel{[(x-x_0)+i(y-y_0)]}}{\cancel{z-z_0}}$$

$$= \left(u_x^0 + iv_x^0\right) + \mathcal{E}$$

where $|\mathcal{E}| \le \dfrac{|R_u|}{|z-z_0|} + \dfrac{|R_v|}{|z-z_0|} \longrightarrow 0$ as $z \to z_0$.

Cor: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$ is analytic on $\mathbb{C}$

entire

Properties of $E(z)$ 0) E is entire

1) $E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} = u + iv = E(z)$

2) $E(0) = 1$

Fun thing: (1), (2) determine $E(z)$!

$$\begin{cases} u_x = u & v_x = v \\ v_y = u & -u_y = v \end{cases}$$

$u(\vec{0}) = 1$, $v(\vec{0}) = 0$ pins down u, v

e.g. $u_x = u \Rightarrow u(x,y) = Ce^x$ $\leftarrow C = C(y)$

3) $E(z+w) = E(z)E(w)$

Group homomorphism: (add grp) $\rightarrow$ (mult grp)

4) $E(-z) = 1/E(z)$ $(w=-z)$

5) $E(z)$ is never zero

6) $E(z-w) = E(z)/E(w)$

7) $E(x) = e^x$, $x \in \mathbb{R}$. $E(z)$ extends e^x, $\mathbb{R}$ to $\mathbb{C}$.

8) $E(i\theta) = \cos\theta + i\sin\theta$

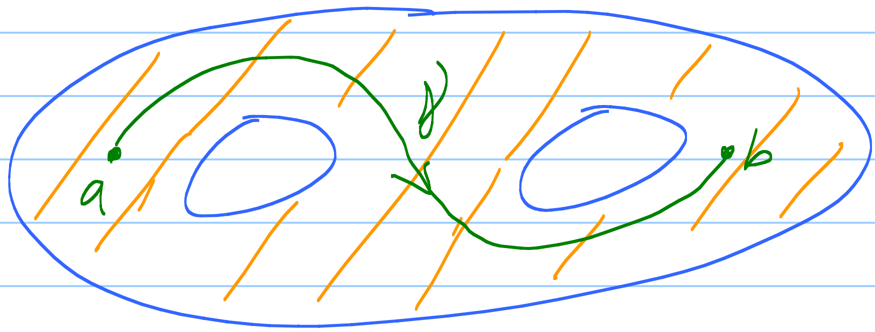
#3 $\Rightarrow$ DeMoivre's.

9) $E(z) = 1 \iff z = 2\pi n i$, $n \in \mathbb{Z}$

Just need to prove #3.

Lemma 1: If f analytic on $D_r(a)$ and $f' \equiv 0$, then f is const.

Wish pf: $f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \equiv 0 \Rightarrow \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$

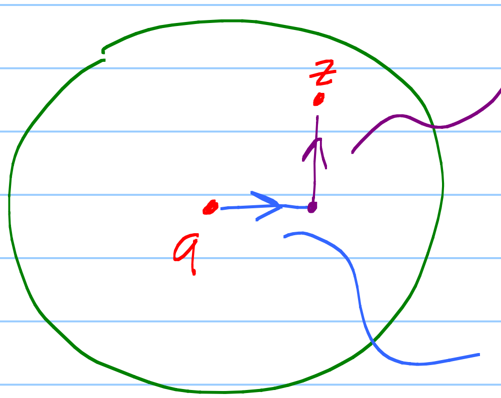


$$\int_\gamma \nabla u \cdot d\vec{s} = u(b) - u(a)$$ ← Ouch! Need to know u C^1-smooth

Freshman calculus: $h(t)$ cont. on $[a,b]$ and diff'ble on (a,b) and $h' \equiv 0$ on (a,b). Then h is const. on $[a,b]$.

Pf: MVT

Pf of Lem 1:



$u_y, v_y \equiv 0$
u, v const on vert

$u_x, v_x \equiv 0$
$\Rightarrow u, v$ const on horiz

Lemma 2: If f is analytic on $D_R(0)$ and

$$f'(z) = k f(z), \quad k \in \mathbb{C} \text{ is a const.}$$

Then $f(z) = C E(kz)$ where $C = f(0)$.

Pf: $f' - \kappa f = 0$. Int. factor: $E(-\kappa z)$

$$E(-\kappa z)f'(z) \underbrace{- \kappa E(-\kappa z)}\, f(z) \equiv 0$$

$$\frac{d}{dz}E(-\kappa z) = \underbrace{E'(-\kappa z)}_{E(-\kappa z)} \underbrace{\frac{d}{dz}[-\kappa z]}_{-\kappa}$$

$$\frac{d}{dz}\left[E(-\kappa z)f(z)\right] \equiv 0$$

So Lem 1 $\Longrightarrow$ $\boxed{E(-\kappa z)f(z) \equiv C}$, const.

Plug in $z=0$ to see that $f(0) = C$.

Stop the proof! Start over using $f(z) = E(\kappa z)$.

Check: $f'(z) = E'(\kappa z)\frac{d}{dz}[\kappa z] = \kappa E(\kappa z) = \kappa f(z)$ ✓

Green box says: $E(-\kappa z)\underbrace{f(z)}_{E(\kappa z)} \equiv \underbrace{C}_{C=f(0)=1}$

$$E(-\kappa z)E(\kappa z) \equiv 1$$

So $E(\kappa z) = 1/E(-\kappa z)$

Back to the proof:

Green box $E(-\kappa z)f(z) = f(0)$ [orig f]

so $f(z) = \dfrac{f(0)}{E(-\kappa z)} = f(0)E(\kappa z)$ ✓

Pf of #3: Let $f(z) = E(z+w)$, w const.

$$f'(z) = E'(z+w)\frac{d}{dz}[z+w]$$

$$= E(z+w) = f(z) \quad !$$

Lemma 2 $\Rightarrow$ $\underbrace{f(z)}_{E(z+w)} = \underbrace{f(0)}_{E(0+w)} E(1\cdot z)$

$$E(z+w) = E(w)E(z) \quad \checkmark$$

Big fact: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$

Way false $\mathbb{R}\to\mathbb{R}$!

implying: f' exists $\Rightarrow$ f cont

f'' exists $\Rightarrow$ f' cont

$\vdots$

All red boxes $\Rightarrow$ u, v are C^∞-smooth.

Assuming big fact (or just that u and v are C^2-smooth), then $f = u + iv$ analytic $\Rightarrow$ u, v harmonic!

Why: C-R eqns
$$\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$$

$\frac{\partial}{\partial x}(A)$: $u_{xx} = v_{xy}$

$+$ $\frac{\partial}{\partial y}(B)$: $u_{yy} = -v_{yx}$

v C^2 smooth $\Rightarrow$ $v_{xy} = v_{yx}$

$\Delta u = u_{xx} + u_{yy} = 0$ u harmonic!

Similarly, $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ shows that v is harmonic.

Big fact: u C^2-smooth and harmonic $\Rightarrow$ u C^∞-smooth.

Problem: Given harmonic u, does there exist harmonic v such $u+iv$ is analytic? [v is called a harmonic conjugate for u.]

Ans: Yes, when Ω has no holes.

How to cook up v:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix}$$

v is a potential fcn for field $\vec{F} = (-u_y, u_x)$