Lecture 9 Analytic functions are given locally by convergent power series

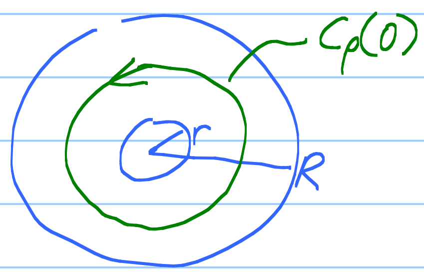
Simple fact: On $D_r(z_0)$, $f_n \twoheadrightarrow f$ $\iff$

$f_n \to f$ <u>uniformly</u> on each $\overline{D_\rho(z_0)}$, $0<\rho<r$. [or $D_\rho(z_0)$]

<u>Why</u>: $|z-z_0|$ continuous on compact $K \subset\subset D_r(z_0)$. $\rho = \max_{z\in K} |z-z_0|$

<u>EX</u>: $f(z)=\frac{1}{z}$ on $\{z: r<|z|<R\}$, $0<r<R$,

has a hole. Pick ρ with $r<\rho<R$.



$H(z)\equiv 1$

$$\int_{C_\rho} \frac{1}{z}\,dz = \int_{C_\rho} \frac{H(z)}{z-0}\,dz$$

$$= 2\pi i\, H(0) = 2\pi i$$ not zero!

Cauchy theorem for Ω with holes will be different.

<u>Notation</u>: $S_N(z) = 1+z+\cdots+z^N$

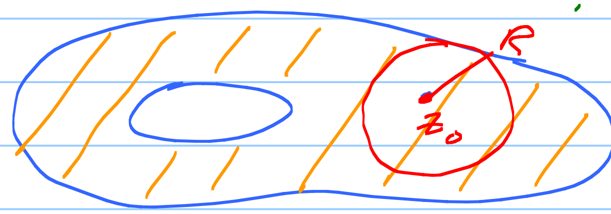
$$\frac{1}{1-z} = S_N(z) + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

Showed that if $|z|<\rho<1$, then

$$|E_N(z)| \le \frac{\rho^{N+1}}{1-\rho}$$

<u>Thm</u>: Suppose f analytic on $D_R(z_0)$. Then f is given by a convergent $\sum_0^\infty a_n (z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$

where R. of C. $\geq R$.

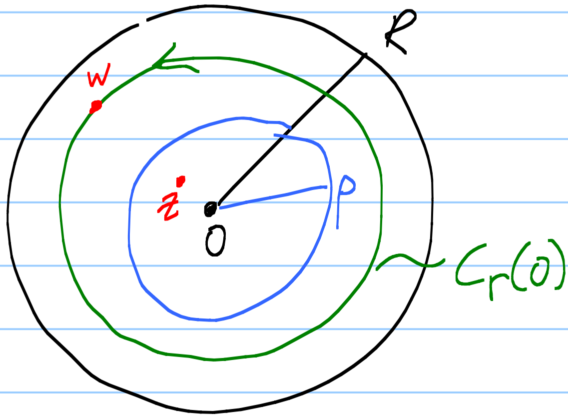


f analytic on Ω

Note: $f(z) = e^z$ or poly, R of $C = \infty$. Maybe $> R$.

Pf: Can assume $z_0 = 0$ (otherwise $w = z - z_0$, etc.)

Let $0 < \rho < r < R$. $\quad |z| < \rho \quad \left|\frac{z}{w}\right| < \frac{\rho}{r} < 1$



$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} \underbrace{\frac{1}{1 - \frac{z}{w}}}_{S_N\left(\frac{z}{w}\right) + E_N\left(\frac{z}{w}\right)} dw$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} \left[1 + \left(\frac{z}{w}\right) + \cdots + \left(\frac{z}{w}\right)^N\right] dw + \underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} E_N\left(\frac{z}{w}\right) dw}_{\mathcal{E}_N(z)}$$

$$= \sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{n+1}}\, dw\right)}_{\frac{f^{(n)}(0)}{n!}} z^n + \mathcal{E}_N(z)$$

and $$\left|\mathcal{E}_N(z)\right| \le \frac{1}{2\pi} \left(\operatorname*{Max}_{C_r} \frac{|f|}{r}\right) \frac{\left(\frac{\rho}{r}\right)^{N+1}}{1 - \frac{\rho}{r}} \underbrace{\text{Length}(C_r)}_{2\pi r}$$

$$\le \frac{M \left(\frac{\rho}{r}\right)^{N+1}}{1 - \frac{\rho}{r}} \quad \text{where } M = \operatorname*{Max}_{C_r} |f|$$

$\longrightarrow 0$ as $N \to \infty$ because $\frac{\rho}{r} < 1$.

So series converges uniformly on $D_\rho(0)$ to f.

Can squeeze ρ, r as close to R as desired.

Get convergence on $D_R(0)$. ✓ So R.of.C $\geq R$

Today: $E(z) = e^z = \sum_0^\infty \frac{z^n}{n!}$, $R = \infty$.

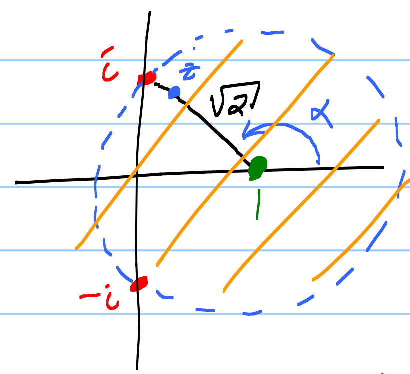
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad R = \infty$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \quad R = \infty$$

$$\tan z = \frac{\sin z}{\cos z} \qquad R = ?$$

EX: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$

$= \sum_0^\infty a_n (z-1)^n$



$z = 1 + re^{i\alpha}$, $\alpha = \frac{3\pi}{4}$

Thm $\Rightarrow$ $R_f \geq \sqrt{2}$

Claim: $R_f \leq \sqrt{2}$ too.

Why: If power series for f converges on $D_\rho(1)$, $\rho > \sqrt{2}$,

it converges to F analytic on $D_\rho(1)$. ← F continuous on $D_\rho(1)$.

$F \equiv f$ on $D_{\sqrt{2}}(1)$. F is cont at i, f is not! ↯.

Thm: Can differentiate power series term by term:

$$f(z) = \sum_0^\infty a_n (z - z_0)^n, \quad R_f > 0.$$

$$f'(z) = \sum_1^\infty n a_n (z-z_0)^{n-1}, \quad R_{f'} = R_f$$

$$\left(= \sum_0^\infty b_n (z-z_0)^n \quad \text{where } b_n = (n+1)a_{n+1} \right)$$

Pf: f' analytic on $D_{R_f}(z_0)$ because f is.

So $R_{f'} \geq R_f$ by Thm. And

$$f'(z) = \sum_0^\infty b_n (z-z_0)^n$$

where $$b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \cdot \frac{n+1}{n+1}$$

$$= (n+1)\frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)a_{n+1}$$

Claim: Know $R_{f'} \geq R_f$. $\leq$ too.

Hmmm. If $R_{f'} > R_f$, the power series for f' would converg to analytic $\tilde{f}$ on $D_{R_{f'}}(0)$. Let F be an antiderivative

Note: $\frac{d}{dz}\left[F - f\right] \equiv 0$ on $D_{R_f}(0)$.

So $F - f \equiv c$, a const on

$f = F - c \leftarrow$ R. of C. $> R_f$ ↯.

Alternatively: Use Hadamard's and $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and LimSup facts.

Next time: Zeroes of analytic functions are "polynomialesque". They are <u>isolated</u> and we can "factor them out."