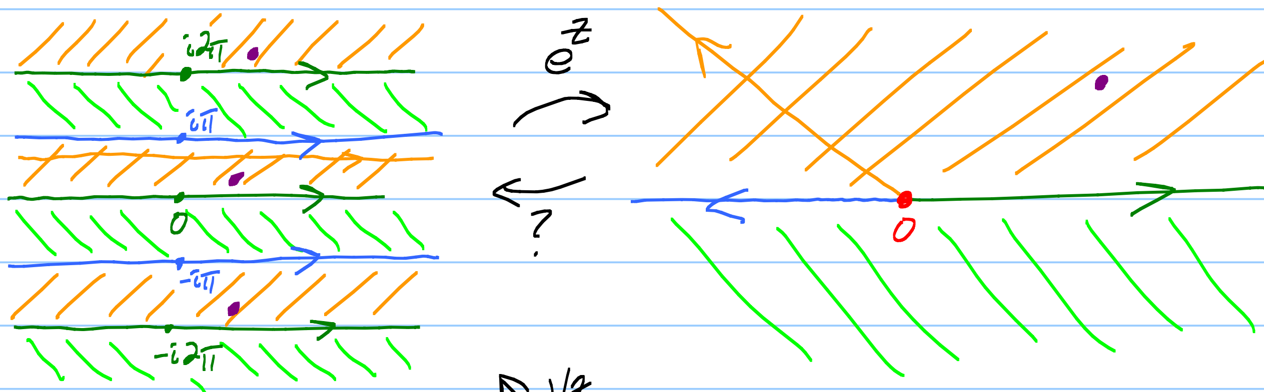
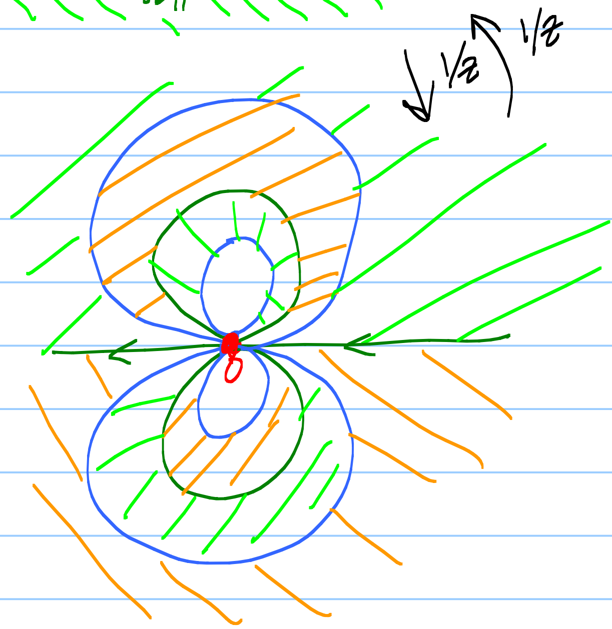
Lecture 13 Isolated singularities & the Riemann removable singularity thm

HWK 4 due next Thurs.



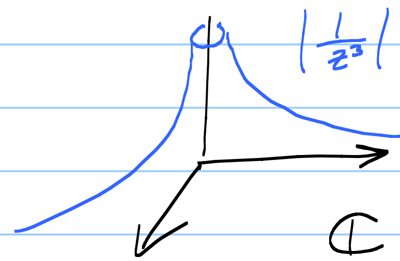


$e^{1/z}$ has a wind blowing singularity at $z=0$! No matter how small $\varepsilon > 0$ is taken, $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ onto $\mathbb{C} - \{0\}$ infinite-to-one!

0 is an "essential singularity" of $e^{1/z}$ at $z=0$.

Other types of behavior: 1) $\frac{1}{z^3}$

$\left|\frac{1}{z^3}\right| = \frac{1}{|z|^3}$



$$\lim_{z\to 0}\left|\frac{1}{z^3}\right| = +\infty$$

$z=0$ is a "pole" of $\frac{1}{z^3}$.

2) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!}}{z} = \frac{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \leftarrow \text{entire!} \quad R = \infty$$

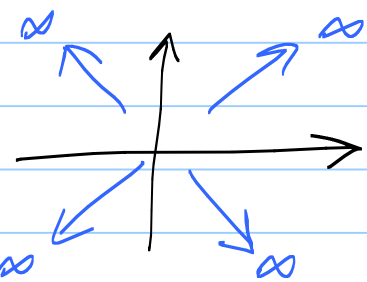
$$f(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases} \quad \text{is entire}$$

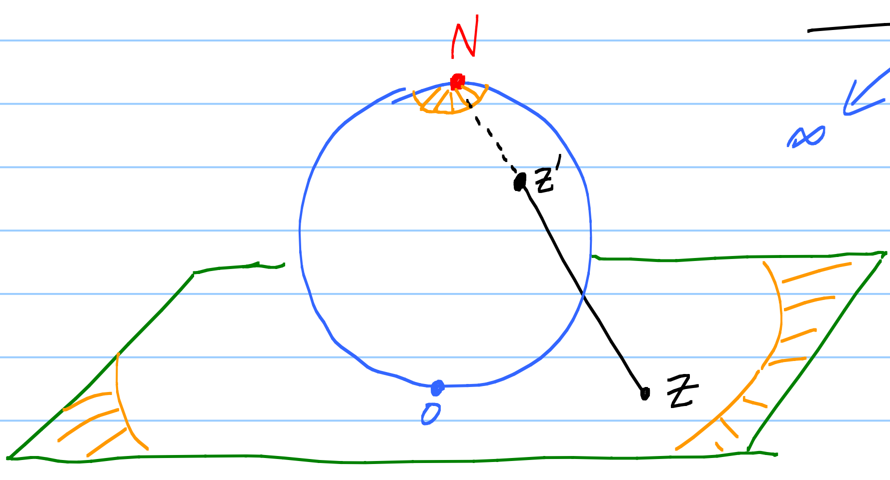
0 is a "removable singularity" for $\frac{\sin z}{z}$.

Big fact: These are the only possible behaviors at an "isolated singularity" (meaning $\exists\ \varepsilon > 0$ such that fcn is analytic on $D_\varepsilon(z_0) - \{z_0\}$).

The "point at infinity": $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$$-\infty \ldots \xrightarrow{\mathbb{R}} +\infty$$





$\mathbb{C}$

$N \sim \infty$

$$D_\varepsilon(\infty) = \{z : |z| > \tfrac{1}{\varepsilon}\} \cup \{\infty\} \leftarrow \text{"polar cap" on Riemann sphere}$$

Def^n: $\lim_{z \to a} f(z) = \infty \leftarrow$ the point at infinity

means, given $M > 0$, there is a $\delta > 0$ such that

$|f(z)| > M$ when $|z-a| < \delta$, $z \neq a$. [Same as $\lim_{z\to a} |f(z)| = +\infty$.]

Think: Setting $f(a) = \infty$ makes f continuous at a as a map to $\hat{\mathbb{C}}$.

Goursat/Morera removable singularity theorem

Suppose f is continuous on a domain Ω and analytic on $\Omega - \{p\}$. Then p is removable. f is analytic on Ω.

Pf: Goursat $\Rightarrow \int_\Delta f\,dz = 0$ for Δ's $\subset \Omega$.

(f cont and $\int_{\Delta\text{'s}} f\,dz = 0$). Morera's $\Rightarrow$ f analytic. ✓

Riemann removable singularity thm: Suppose f is analytic on $D_\rho(a) - \{a\}$ and bounded there (meaning $\exists\, M > 0$ such that $|f(z)| < M$ on $D_\rho(a) - \{a\}$.)

Then f can be given a value at a that makes it analytic on $D_\rho(a)$, i.e., a is removable.

Way false $\mathbb{R} \to \mathbb{R}$: $\sin(1/t)$

Pf: Define
$$G(z) = \begin{cases} f(z)(z-a) & z \neq a \\ 0 & z = a \end{cases}$$

Note: $\lim_{z\to a} G(z) = 0$ because $|f| < M$ $\left[\begin{array}{l} \varepsilon > 0 \\ \delta = \frac{\varepsilon}{M} \end{array}\right.$ ✓

G analytic on $D_\rho(a)-\{a\}$.

Goursat / Morera rem sing thm $\Rightarrow$ G analytic on $D_\rho(a)$.

$$\text{So} \quad G(z) = \underbrace{G(a)}_{=0} + G'(a)(z-a) + \frac{G''(a)}{2!}(z-a)^2 + \cdots$$

$$= (z-a)\left[\underbrace{G'(a) + \frac{G''(a)}{2!}(z-a)^2 + \cdots}_{g(z),\ \text{R. of C.} \geq \rho}\right]$$

$$f(z) = \begin{cases} \frac{G(z)}{z-a} = g(z), & z \neq a \\ g(a) = G'(a) & \text{should be value at } z=a. \end{cases}$$

Alternate pf: Let $H(z) = \begin{cases} f(z)(z-a)^2 & z \neq a \\ 0 & z = a \end{cases}$

Show that $DQ = \frac{H(z)-0}{z-a} \longrightarrow 0$ as $z \to a$.

So H is analytic on $D_\rho(a)$. Have $H(a) = 0$, $H'(a) = 0$.

$$\text{So} \quad H(z) = \frac{H''(a)}{2!}(z-a)^2 + \cdots \quad \text{etc.}$$

Def^n: Suppose a is an isolated sing of f analytic on $D_\rho(a)-\{a\}$.

1) a is "removable" if f can be given a value at a that makes it analytic on $D_\rho(a)$.

2) a is a "pole" if $\lim_{z\to a} f(z) = \infty$

3) a is an "essential" if, for every ε with $0 < \varepsilon < \rho$, $f(D_\varepsilon(a) - \{a\})$ is dense in $\mathbb{C}$,

meaning, every pt in $\mathbb{C}$ is a limit pt

$\{w : w = f(z) \text{ for some } z \in D_\varepsilon(a) - \{a\}\}$

Thm: These are the only 3 possible behaviors.

Important map: Outside of a little disc to $D_1(0)$