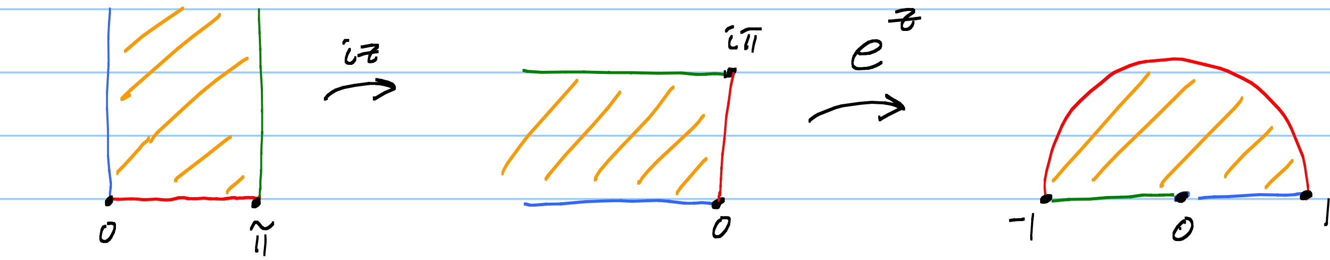
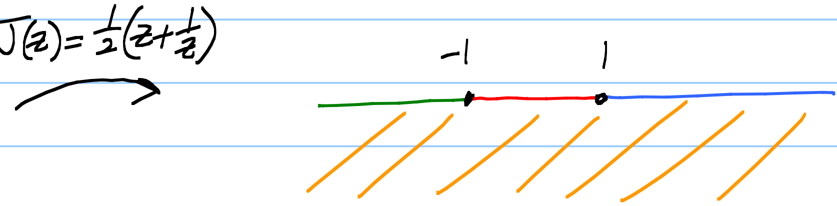
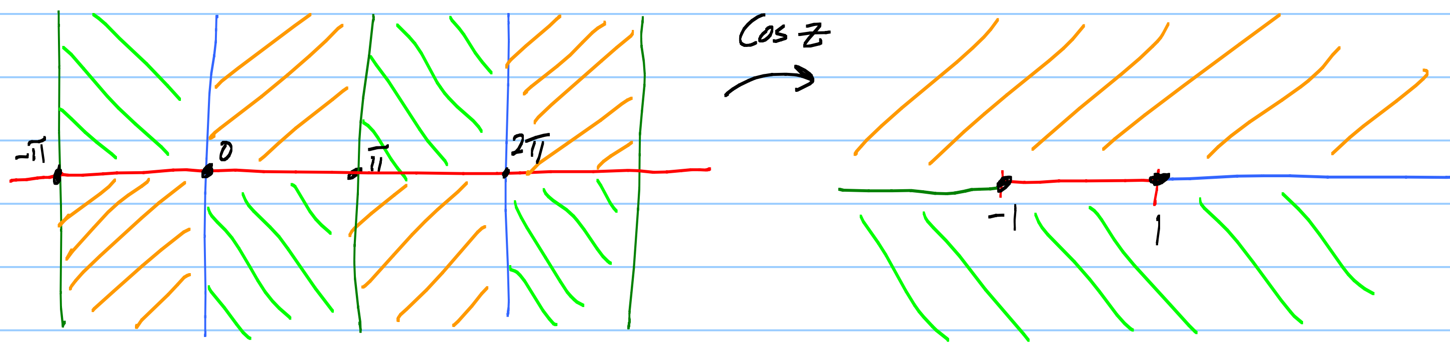
Jukovsky map & $\cos z$





Composition $= \cos z$



$\cos(-z) = \cos z$

$\cos(z+\pi) = -\cos z$

Partial fractions: $\dfrac{p(z)}{q(z)}$ ← factor out common factors

If $\deg p \geq \deg q$, first do long division:

$$\frac{p}{q} = (\text{Poly}) + \frac{r}{q} \quad \text{where } \deg r < \deg q$$

Partial fractions decomp thm: If P, Q polys with no common factors and $N_P = \deg P < \deg Q = N_Q$, then $\frac{P}{Q}$ has a part. frac. decomp as follows:

$$Q(z) = A\,(z-r_1)^{m_1}(z-r_2)^{m_2}\cdots(z-r_n)^{m_n}$$

where $\sum_1^n m_k = N_Q$.

Near r_k: $$\frac{P(z)}{Q(z)} = \frac{1}{(z-r_k)^{m_k}}\Big[\underbrace{\frac{P(z)}{q_k(z)}}_{\substack{A_0 + A_1(z-r_k)+\cdots \\ \text{analytic near } r_k}}\Big] \leftarrow q_k(r_k) \neq 0$$

$$= \underbrace{\frac{A_0}{(z-r_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-r_k}}_{R_k = \text{Principal part at } r_k} + \left(\begin{array}{c}\text{analytic} \\ \text{near } r_k\end{array}\right)$$

Claim: $\frac{P}{Q} = \sum_{k=1}^n R_k \leftarrow$ par. frac. decomp.

Let $f = \frac{P}{Q} - \sum_1^n R_k$

Claim: f is bndd entire.

Near r_j: $$f = \underbrace{\left(\frac{P}{Q} - R_j\right)}_{\substack{\text{removable sing} \\ \text{at } r_j}} - \underbrace{\sum_{k\neq j} R_k}_{\substack{\text{analytic} \\ \text{near } r_j}}$$

Aha! Can give f values at r_j's to make it entire.

Claim: $f \to 0$ as $z \to \infty$.

Note: $\lim_{z\to\infty} R_j(z) = 0$. ✓

Basic poly est. : $a|z|^{N_P} \le |P(z)| \le A|z|^{N_P}$, $|z| > R_P$

$b|z|^{N_Q} \le |Q(z)| \le B|z|^{N_Q}$, $|z| > R_Q$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A|z|^{N_P}}{b|z|^{N_Q}} = \frac{A}{b}\frac{1}{|z|^{N_Q - N_P}} \text{ if } |z| > \mathrm{Max}(R_P, R_Q)$$

$$\longrightarrow 0 \text{ as } z \to \infty.$$

[$|f| < 1$ outside $D_R(0)$. $|f|$ cont on $\overline{D_R(0)}$.

See $|f|$ bdd on $\mathbb{C}$.]

Liouville's $\Rightarrow f \equiv c$, const. c must $= 0$. ✓

Freshman calc: $$\frac{x^3 + 7x + 13}{(x-2)^2(x^2+x+1)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \underbrace{\frac{Cx+D}{x^2+x+1}}_{\frac{E}{x-r} + \frac{\bar{E}}{x-\bar{r}}}$$

Schwarz lemma (Chap 8, sec 2)

Suppose $f: D_1(0) \to D_1(0)$ is analytic

$(|f(z)| < 1$ when $|z| < 1)$, and $\boxed{f(0) = 0}$.

Then A) $|f(z)| \le |z|$

B) $|f'(0)| \le 1$

and if equality holds in (A) for some $z \ne 0$, or

if equality holds in (B), then

$f(z) = \lambda z$ for some unimodular const $\lambda = e^{i\theta}$.

Pf: Let $F(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ $\quad DQ = \frac{f(z) - \overbrace{f(0)}^{0}}{z - 0}$

F analytic on $D_1(0) - \{0\}$ and cont. on $D_1(0)$.

So 0 is a removable sing.

Let $z_0 \in D_1(0)$. Pick r with $|z_0| < r < 1$.

Max Princ: $$|F(z_0)| \leq \max_{|z|=r} |F| = \max_{|z|=r} \frac{|f(z)|}{r} < \frac{1}{r}$$

Aha! Can let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$.

Conclude that $|F(z_0)| \leq 1$. z_0 arbitrary!

$$|F(z)| = \begin{cases} \frac{|f(z)|}{|z|} & z \neq 0 \quad (A) \\ |f'(0)| & z = 0 \quad (B) \end{cases} \qquad f(0) = 0$$

Part 2 If a max of $|F|$ $(= 1)$ at a point in $D_1(0)$, then $F \equiv c$, const. $|c| = 1$.

$\frac{f(z)}{z} = c = e^{i\theta}$, $z \neq 0$. $f(z) = e^{i\theta} z$, $z \neq 0$.

true at $z = 0$ too.

Log on a convex open set

Given a nonvanishing analytic fcn F on a convex open Ω, $\exists$ analytic G on Ω such that $e^G = F$ on Ω. So $H = e^{G/N}$ is an analytic N-th root of F.

Idea: If $e^G = F$

$$\frac{d}{dz}\left(e^G\right) = F'$$

$$G' \underbrace{e^G}_{F} = F'$$

$$G' = \frac{F'}{F}$$

Idea: Define

$$G(z) = \int_{[a,z]} \frac{F'(w)}{F(w)}\,dw$$

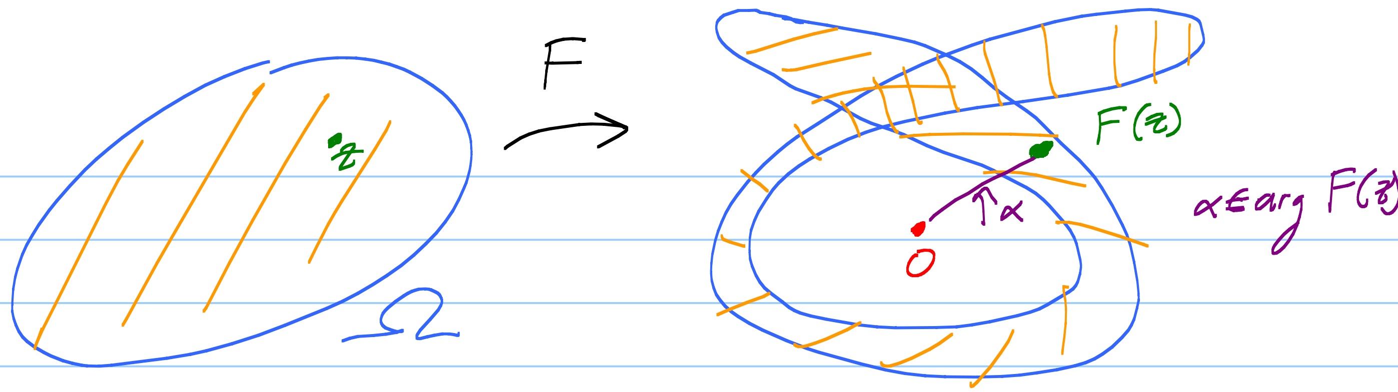
Then $G' = \frac{F'}{F}$. Is $e^G = F$? Almost!

Trick: $$\frac{d}{dz}\left(\frac{F}{e^G}\right) = \frac{F' e^G - \overset{F'/F}{G'} e^G F}{\left(e^G\right)^2} \equiv 0$$

So $\frac{F}{e^G} \equiv c$, const.

$$F = c\,e^G = e^{\operatorname{Log} c}\, e^G = e^{\underbrace{\operatorname{Log} c + G}_{\text{that's my } G!}}$$

Pic:



$$e^G = F \qquad \operatorname{Re} G = \operatorname{Ln}|F|$$

$$\operatorname{Im} G \in \arg F$$

$$G(z) = \operatorname{Ln}|F| + i\theta(z) \longleftarrow \text{continuous choice of } \arg F$$

Two G's on a domain:

$$G_1 \equiv G_2 + i2\pi n \quad n \in \mathbb{Z} \quad \text{on } \Omega,$$

Two N-th roots on a domain:

$$H_1 \equiv \lambda H_2 \qquad \lambda \ N\text{-th root of unity}$$