Lecture 16 The Open Mapping Theorem

HWK 4 due tomorrow in GS 1

OMT: Non-constant analytic fcns map open sets to open sets.

Way false $\mathbb{R} \to \mathbb{R}$: $h(t) = 1 - t^2$. $h(\underbrace{(-1,1)}_{\text{open}}) = (0,1]$ ← not open

Assume f analytic on a domain Ω.

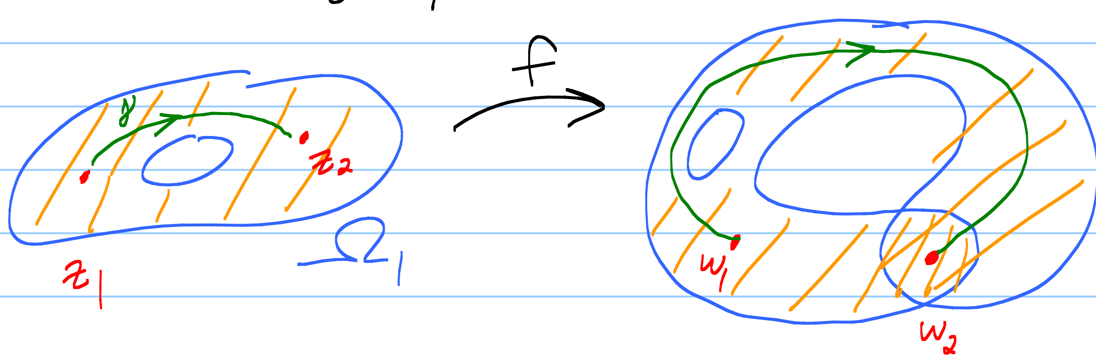
Notation: $f(U) = \{f(z) : z \in U\}$

$f^{-1}(U) = \{z : f(z) \in U\}$

Fact: f continuous $\Longleftrightarrow$ $f^{-1}(U)$ is open when U is open.

Cor of OMT: $f : \Omega_1 \to \Omega_2$ analytic on a domain Ω_1, $\Omega_2 = f(\Omega_1)$. Then Ω_2 is also a domain.

Pf: OMT $\Rightarrow$ Ω_2 open. ✓



$f(\gamma)$ connects w_1 to w_2. Ω_2 connected ✓

Cor of OMT: $f : \Omega_1 \xrightarrow[\text{onto}]{1-1} \Omega_2$ analytic.

1) Then $F = f^{-1}$ is continuous on Ω_2.

Pf: $F^{-1}(U) = f(U)$ open when U open ✓

2) (1) $\Rightarrow$ F is analytic!

Pf: $w = f(z)$ $\quad F(w) = z$

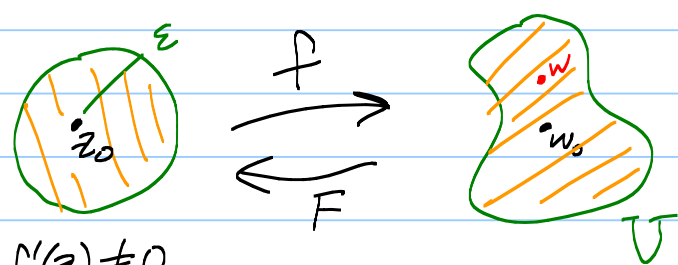
$w_0 = f(z_0)$ $\quad F(w_0) = z_0$

F cont. $\Longrightarrow$ $\underbrace{F(w)}_{z} \to \underbrace{F(w_0)}_{z_0}$ as $w \to w_0$

$$DQ = \frac{F(w)-F(w_0)}{w-w_0} = \frac{z-z_0}{f(z)-f(z_0)} \longrightarrow \frac{1}{f'(z_0)} \text{ as } w \to w_0,$$

provided that $f'(z_0) \neq 0$. $\quad$ [$F'(w_0)$ exists when $f'(z_0) \neq 0$.]

Hmmm. f not const. So $f' \not\equiv 0$. So zeroes of f' are isolated. Say $f'(z_0) = 0$.



$f'(z) \neq 0$ if $z \in \overline{D_\varepsilon(z_0)} - \{z_0\}$

F' exists and is nonvanishing on $U - \{w_0\}$. F is cont on U, w_0 is a removable sing of F!

Aha! $\quad F(f(z)) = z$

$F'(f(z))f'(z) = 1$ $\longleftarrow$ neither f nor F can have a vanishing derivative.

Fact: Locally 1-1 analytic fcns have nonvanishing derivatives.

Way false $\mathbb{R} \to \mathbb{R}$: $\underbrace{h(t) = t^3}_{C^\infty \text{ smooth}}$: $(-1,1) \xrightarrow[\text{onto}]{1-1} (-1,1)$

but $h^{-1}(s) = s^{1/3}$ is not diff'ble at $s=0$!

Argument principle for a disc. Suppose f is analytic on $D_R(z_0)$ and $0 < r < R$ and suppose has no zeroes on $C_r(z_0) = \{z : |z - z_0| = r\}$. Then

$$\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)}\, dz = \begin{pmatrix} \#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity} \end{pmatrix}$$

<u>Lemma</u>: Suppose f has a zero of order N at a. Then the principal part of $\frac{f'}{f}$ at a is $\frac{N}{z-a}$.

<u>Pf</u>: Know $f(z) = (z-a)^N F(z)$ where F analytic, $F(a) \neq 0$.

$$f'(z) = N(z-a)^{N-1} F(z) + (z-a)^N F'(z)$$

So

$$\frac{f'(z)}{f(z)} = \underbrace{\frac{N}{z-a}}_{\text{P.P.} \checkmark} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } a}$$

<u>Pf of Arg princ</u>: Let $a_1, \ldots, a_N$ be the zeroes of f in $\overline{D_r(z_0)}$. [Only <u>finitely</u> many zeroes: If there were ∞ many zeroes in $\overline{D_r(z_0)}$ $\leftarrow$ compact. They'd have a limit pt in $\overline{D_r(z_0)}$. f <u>not</u> zero on $C_r(z_0)$. So lim pt in $D_r(z_0)$. $\Longrightarrow f \equiv 0$. ↯.]

Let m_j = order of zero at a_j.

$$\frac{f'}{f} - \sum_{1}^{N} \frac{m_j}{z - a_j} = G(z) \quad \text{has \underline{removable} sing at } a_j\text{'s.}$$

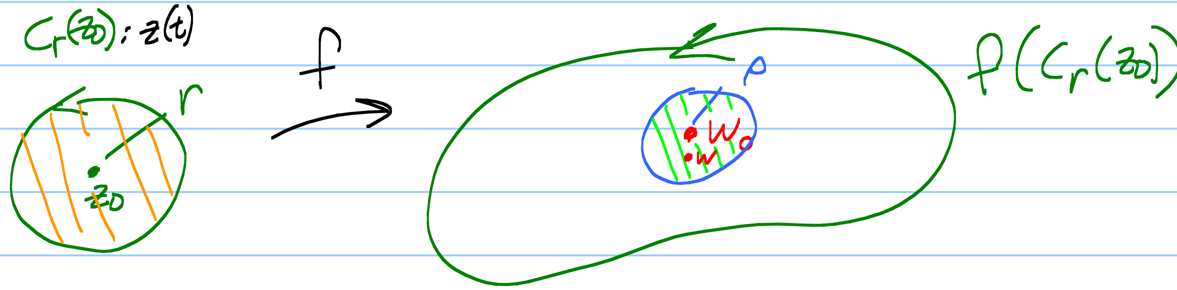
Give G the values at a_j to make it analytic.

$$\int_{C_r(z_0)} \frac{f'}{f}\,dz = \sum_1^N m_j \underbrace{\int_{C_r(z_0)} \frac{1}{z-a_j}\,dz}_{2\pi i} + \underbrace{\int_{C_r(z_0)} G\,dz}_{=0}$$

by Cauchy Thm on convex

$\sum_1^N m_j$ = total # zeroes, counted with mult. ✓

Pf of OMT: f analytic on a domain Ω and non-constant. Pick $w_0 \in f(\Omega)$. Then $\exists\, z_0 \in \Omega$ with $f(z_0) = w_0$. f not constant, so $f(z) - w_0$ has an isolated zero at z_0. So $\exists\, r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$ and z_0 is the only zero of $f(z) - w_0$ in $\overline{D_r(z_0)}$.



Since $|f(z(t)) - w_0|$ is cont. and non-vanishing, it has a min $= m > 0$. Pick ρ with $0 < \rho < m$.

Claim: Every $w \in D_\rho(w_0)$ gets hit by f on $D_r(z_0)$.

$$\begin{pmatrix} \#\text{ zeroes of } f(z)-w \\ \text{inside } C_r(z_0) \end{pmatrix} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)-w}\,dz = H(w)$$

Aha! Function of w on right is analytic!

Fact: A continuous integer valued fcn on a domain must be constant!

Pf that H is analytic:

$$H(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(z(t))}{f(z(t))-w}\, z'(t)\, dt$$

$f(C_r(z_0))$: $\zeta(t) = f(z(t))$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\zeta'(t)}{\zeta(t)-w}\, dt$$

$$= \frac{1}{2\pi i}\int_{f(C_r(z_0))} \frac{1}{\zeta - w}\, d\zeta$$

HWK 2 prob 1.

$H(w_0) = \#$ zeroes of $f(z)-w_0$ in $C_r(z_0)$ ≥ 1.

So $H(w) \geq 1$ too for $w \in D_\rho(w_0)$ [= same # !]

So w gets hit on $D_r(z_0)$. Pf of OMT is done.