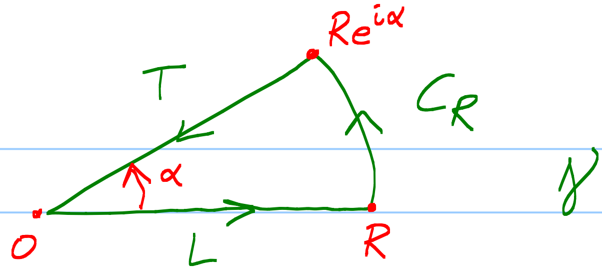
Lecture 22 Important consequences of the Schwarz lemma

HWK 5 due Thurs (tomorrow)

EX $I = \displaystyle\int_0^\infty \underbrace{\frac{1}{x^3+1}}_{f(z)=\frac{1}{z^3+1}}\,dx$



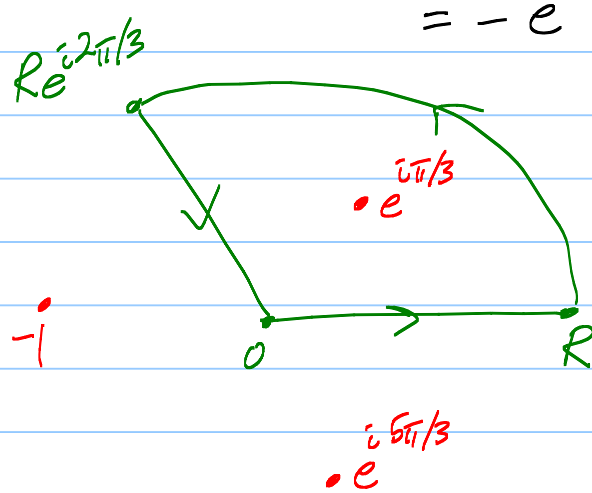
$$\int_L f\,dz = \int_0^R \frac{1}{t^3+1}\,dt \leftarrow \text{want}, \qquad \left|\int_{C_R} \frac{1}{z^3+1}\,dz\right| \le \frac{1}{R^3-1}\cdot \alpha\pi \to 0 \text{ as } R\to\infty$$

$$T:\quad -T:\ z(t) = te^{i\alpha},\quad 0\le t\le R,\quad z'(t) = e^{i\alpha}.$$

$$\int_T f\,dz = -\int_{-T} f\,dz = -\int_0^R \frac{1}{\underbrace{(te^{i\alpha})^3+1}_{t^3e^{i3\alpha}}}\cdot e^{i\alpha}\,dt$$

$\leftarrow$ Aha! Want $3\alpha = 2\pi$, $\alpha = \frac{2\pi}{3}$

$$= -e^{i2\pi/3}\int_0^R \frac{1}{t^3+1}\,dt$$



$$z^3+1 = 0$$
$$z = e^{i\pi/3},\ -1,\ e^{i5\pi/3}$$

$$\left(\int_L + \int_T\right) + \int_{C_R} f\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/3}} \frac{1}{z^3+1}$$

$$\downarrow \qquad \downarrow \qquad\qquad \downarrow$$

$$\left(1-e^{i2\pi/3}\right)I + 0 = 2\pi i\,\frac{1}{3\left(e^{i\pi/3}\right)^2}$$

$$I = \frac{2\pi i}{3e^{i2\pi/3}\left(1-e^{i2\pi/3}\right)}$$

$$= \frac{\pi}{3} \cdot \frac{2i}{1-e^{i2\pi/3}} e^{-i2\pi/3} \cdot \left(\frac{-e^{-i\pi/3}}{-e^{-i\pi/3}} \right)$$

$$= \frac{\pi}{3} \quad \frac{2i}{e^{i\pi/3} - e^{-i\pi/3}} \underbrace{\left(-e^{-i3\pi/3}\right)}_{-(-1)}$$

$$= \frac{\pi/3}{\sin \pi/3}$$

Applications of the Schwarz lemma

Hwk 1: prob. 1 : $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ $a \in D_1(0)$.

Möbius transformation

Facts: φ_a analytic, $\varphi_a : D_1(0) \circlearrowleft$ 1-1 onto

$$\varphi_a^{-1} = \varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z}$$

Def^n: $\mathrm{Aut}(\Omega) = \{ f : \Omega \circlearrowleft \text{ 1-1 onto, analytic} \}$

domain

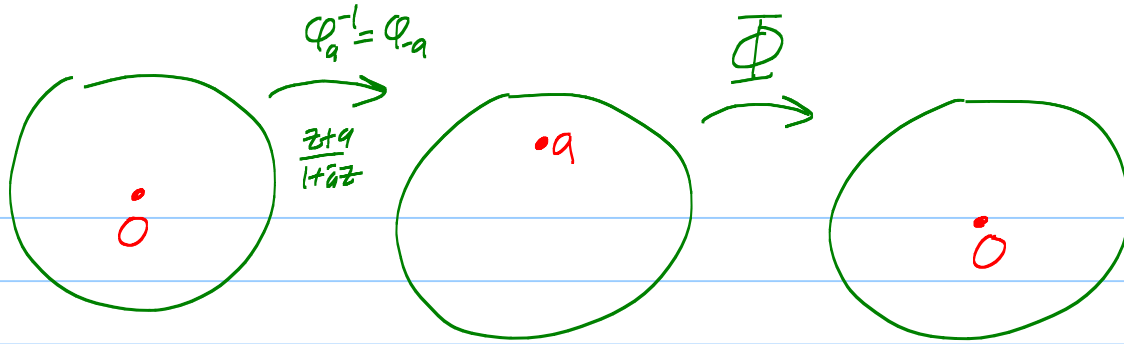
biholomorphic self map of Ω

Fact: $\mathrm{Aut}(\Omega)$ is a group under composition

Pf: Super inverse function theorem: $(f^{-1}) \in \mathrm{Aut}(\Omega)$ when $f \in \mathrm{Aut}(\Omega)$.

Thm: $\mathrm{Aut}(D_1(0)) = \{ \lambda \varphi_a : a \in D_1(0), |\lambda| = 1 \}$

Pf: Suppose $\Phi \in \mathrm{Aut}(D_1(0))$



$$\Psi = \Phi(\varphi_{-a})$$

Lemma: If $\Psi \in \mathrm{Aut}(D_1(0))$ and $\Psi(0) = 0$, then

$\Psi(z) = \lambda z$ for some unimodular const λ.

Pf: Schwarz $\Rightarrow$ $|\Psi'(0)| \le 1$.

But Ψ^{-1} satisfies same hyp! (Super IFT too)

So $\underbrace{|(\Psi^{-1})'(0)|}_{\frac{1}{|\Psi'(0)|}} \le 1$

$\Rightarrow |\Psi'(0)| \ge 1$

So $|\Psi'(0)| = 1$ and Schwarz $\Rightarrow \Psi(z) = \lambda z$ ✓

Back to proof: Now have

$$\Phi(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda \underbrace{z}_{\varphi_a(w)}$$

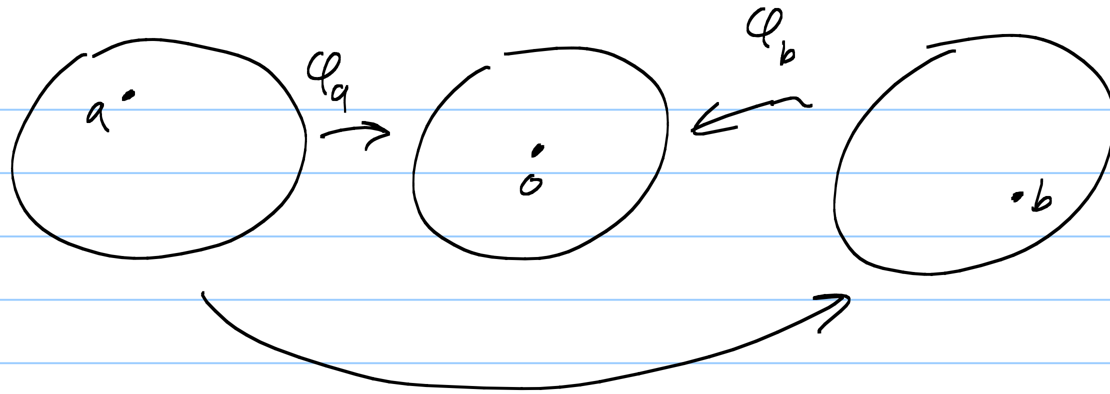
$w = \varphi_{-a}(z)$

$z = (\varphi_{-a})^{-1}(w)$

$= \varphi_a(w)$

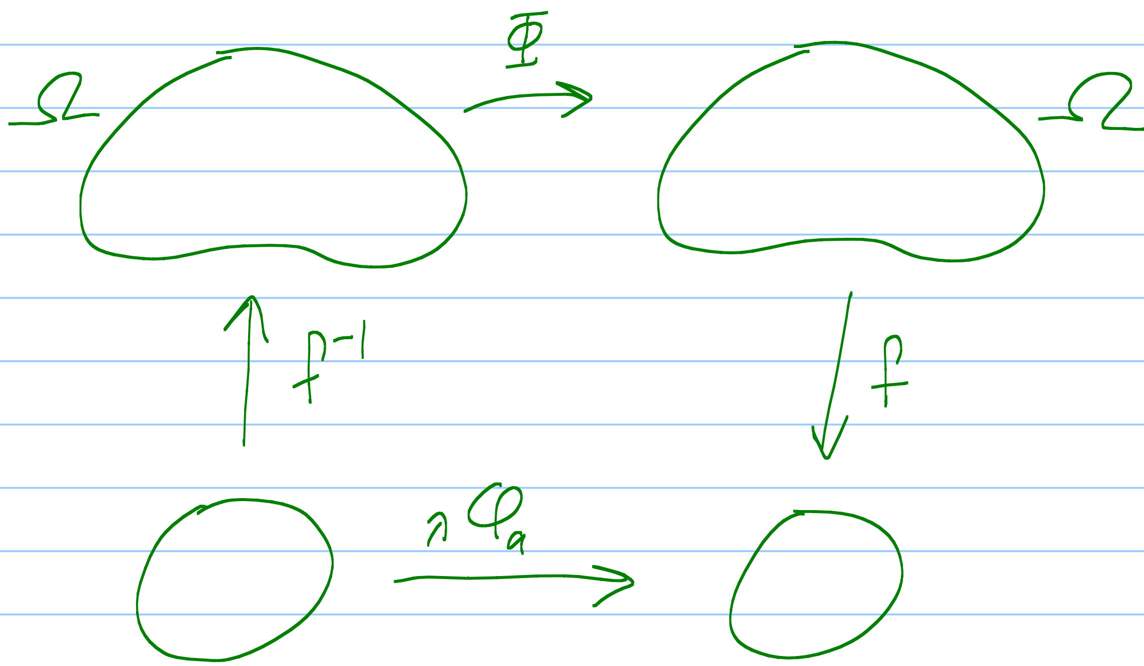
So $\Phi(w) = \lambda \varphi_a(w)$ ✓

Fact: $\mathrm{Aut}(D_1(0))$ is transitive

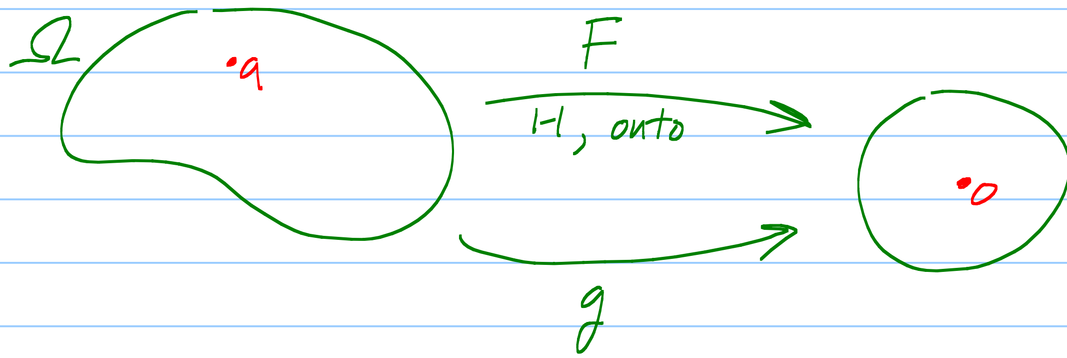


$$\varphi_b^{-1} \circ \varphi_a = \varphi_{-b} \circ \varphi_a$$

Riemann mapping thm: $\Omega \not\subset \mathbb{C}$, simply connected (no holes), $\exists\, f : \Omega \to D_1(0)$ 1-1, onto, analytic.



Important HWK 5 problem:



F, g analyt.

$$\begin{cases} F(a) = 0 \\ g(a) = 0 \end{cases}$$

Schwarz : $|g'(a)| \le |F'(a)|$

$M = \sup\{|g'(a)| : g: \Omega \to D_1(0) \text{ analytic}, g(a)=0\}$

Riemann map F : $|F'(a)| = M$.

Strategy to find F:

Step 1: $M < \infty$. (Cauchy estimates)

Step 2: Get seq $g_n : \Omega \to D_1(0)$, $g_n(a) = 0$ analytic, $|g_n'(a)| \to M$.

Step 3: Show $\exists$ subseq of $\{g_n\}$ that converges uniformly on compact sets of Ω to f. (Montel's Thm)

Know f is analytic.

Step 4: Show $f: \Omega \to D_1(0)$ is 1-1. (Hurwicz)

Step 5: Show f onto. (Schwarz lemma)

f is Riemann map!

Linear fractional transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0$$

Some books: LFTs = Möbius transf, not here.

<u>Facts</u>: $L : \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1\text{-}1} \hat{\mathbb{C}}$

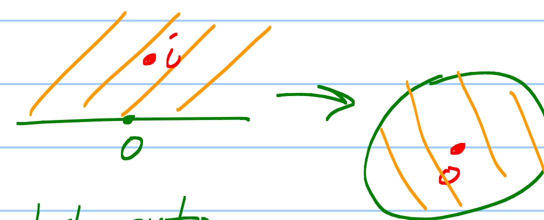
$$\lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} \qquad L(\infty) = \frac{a}{c}$$

(If $c=0$, then $d \neq 0$. $L(z) = Az+B$ $\quad L(\infty) = \infty$.)

$z = \frac{-d}{c}$ is a simple pole, $c \neq 0$.

Think $L\left(\frac{-d}{c}\right) = \infty$

$\mathrm{Aut}(\mathrm{UHP}) = ?$ $\qquad \frac{z-i}{z+i}$:



1-1, onto

Big fact: LFTs map $\begin{Bmatrix} \text{lines,} \\ \text{circles} \end{Bmatrix} \to \begin{Bmatrix} \text{lines,} \\ \text{circles} \end{Bmatrix}$