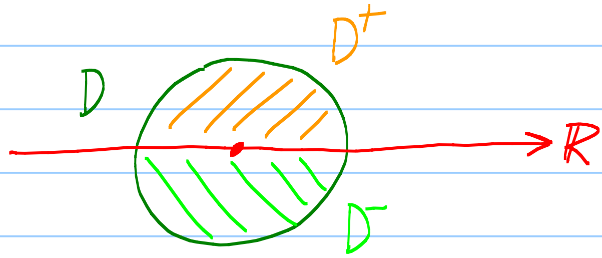
Lecture 25 Schwarz reflection

HWK 6 due Thurs in GS

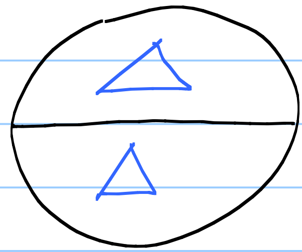
Lemma



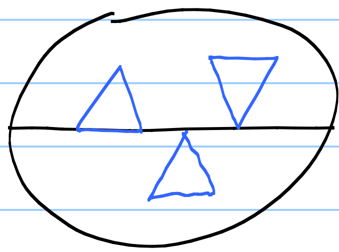
f continuous on D, analytic in D^+ and D^-.
open

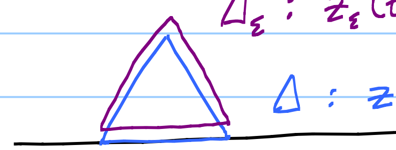
Then f analytic on D.

Pf:



$\int_\Delta f\,dz = 0$ when $\Delta \subset D^+$ or D^-.



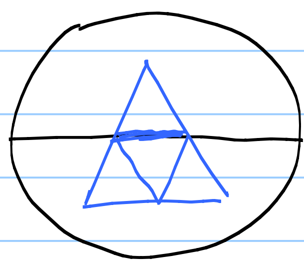


$\Delta_\varepsilon : z_\varepsilon(t) = z(t) + i\varepsilon$

$\Delta : z(t)$

$$\int_\Delta f\,dz = \lim_{\varepsilon \to 0^+} \underbrace{\int_{\Delta_\varepsilon} f\,dz}_{=0}$$

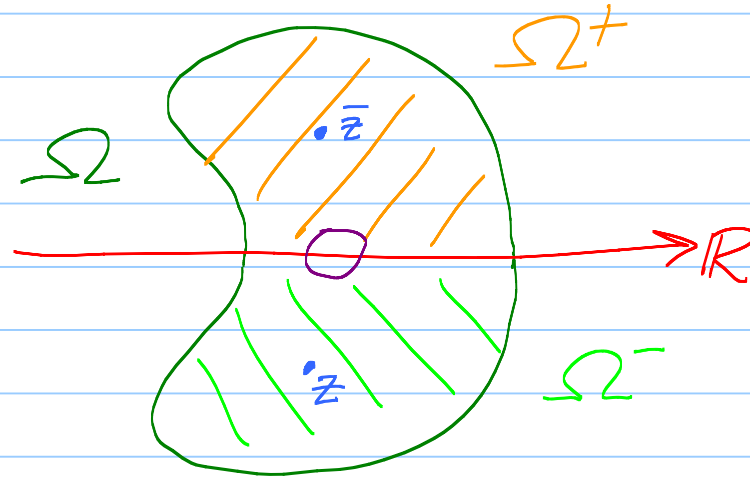
Goursat trick! →



Aha $\int_\Delta f\,dz$ for any $\Delta \subset D$.

Morera's $\Rightarrow$ f analytic.

Schwarz reflection principle



Ω symmetric domain about real line

1) f continuous on Ω^+ up to R,

2) $f(x) \in \mathbb{R}$ for $x \in \Omega \cap \mathbb{R}$, and

3) f is analytic in Ω^+.

Then
$$F(z) = \begin{cases} f(z) & \text{when } z \in \Omega^+ \\ f(x) & \text{when } x \in \Omega \cap \mathbb{R} \\ \overline{f(\bar{z})} & \text{when } z \in \Omega^- \end{cases}$$

is <u>analytic</u> on Ω.

<u>Pf</u>:- HYP $\Rightarrow$ F continuous on Ω. ✓
F analytic on Ω^+. ✓

HWK 5 prob: $\overline{f(\bar{z})}$ is analytic on Ω^-. ✓

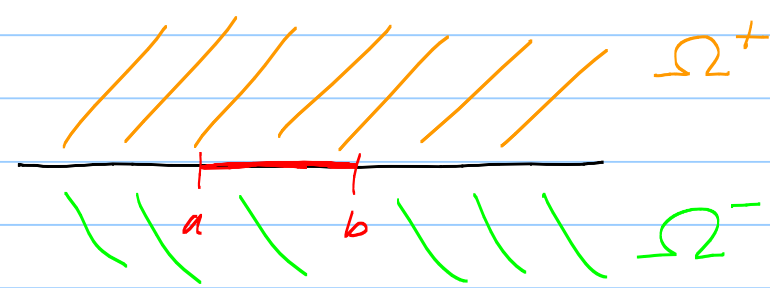
Lemma $\Rightarrow$ F analytic on Ω.

HWK 5 prob: $f(x+iy) = u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\overline{f(x-iy)} = \underbrace{u(x,-y) - iv(x,-y)}_{\text{C-R eqns + Chain rule. ✓}} = \underbrace{\overline{\sum_{n=0}^{\infty} a_n (\bar{z}-z_0)^n}}_{\text{Same R of C}}$$

<u>Application of Schwarz</u> Suppose f is analytic on UHP and continuous up to $(a,b) \subset \mathbb{R}$ and <u>zero</u> there.

Then $f \equiv 0$ on UHP.

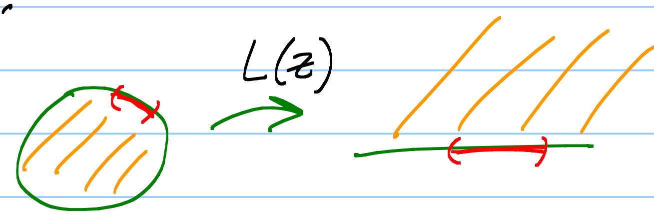


Schwarz $\Rightarrow f$ extends analytic to $\mathbb{C} - \left[(-\infty, a] \cup [b, \infty)\right]$

Every pt in (a,b) is a limit pt of $(a,b) \subset \Omega$.

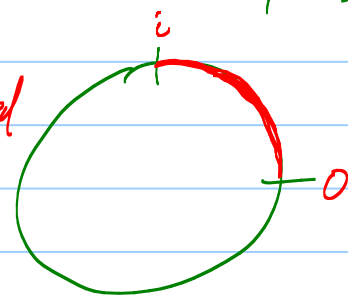
Identity thm $\Rightarrow f \equiv 0$.

Example true for discs too.



Riemann mapping thm can be used to generalize.

<u>Fun exercise</u> f continuous on $\overline{D_1(0)}$, analytic in $D_1(0)$ and zero on red



Then $f \equiv 0$.

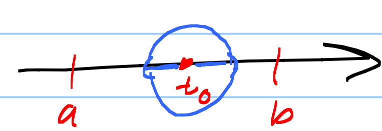
<u>Pf</u>: $F(z) = f(z) f(iz) f(-z) f(-iz)$ is cont on $\overline{D_1(0)}$, analytic in $D_1(0)$, $\equiv 0$ on $C_1(0)$.

Max princ $\Rightarrow F \equiv 0$ on $D_1(0)$

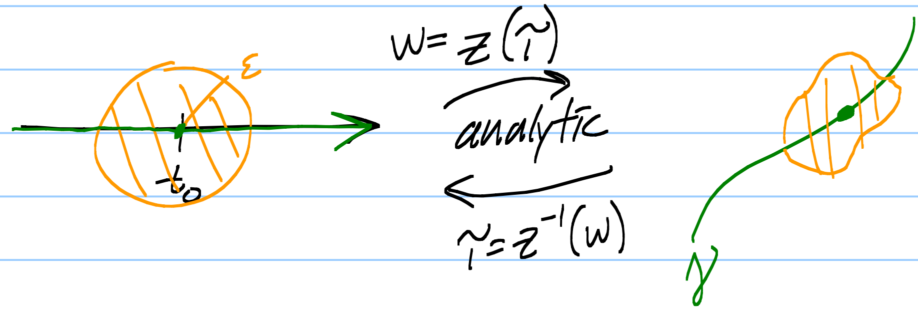
$\Rightarrow$ one of factors $\equiv 0 \Rightarrow f \equiv 0$.

Why Schwarz reflection is important:

"Real analytic" curve γ: $z(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$



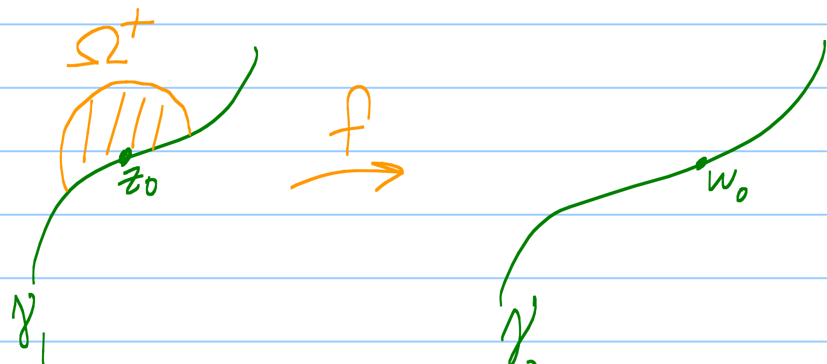
real conv power series $\Longrightarrow$ complex conv power series on $D_\varepsilon(t_0)$. Let $\tilde{\tau}$ be complex near t_0



Assume $z'(t_0) \neq 0$. Then $z'(\tilde{\tau})$ nonvanishing near t_0.

So can shrink ε so $z(\tilde{\tau})$ is 1-1 onto.

Cor:



real analytic curves

f analytic on Ω^+ and continuous up to γ_1.

If $f(\gamma_1) \subset \gamma_2$, then f extends to be analytic on a nbhd of z_0.

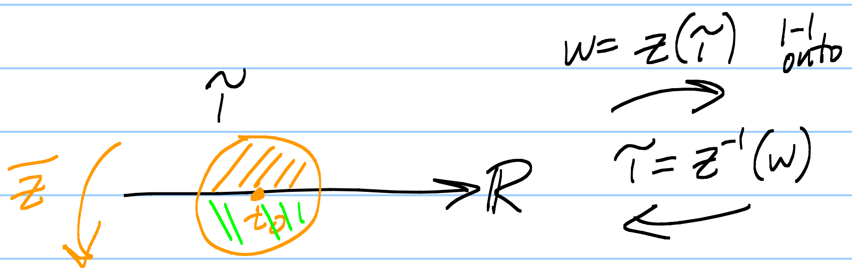
Schwarz reflection fcns:

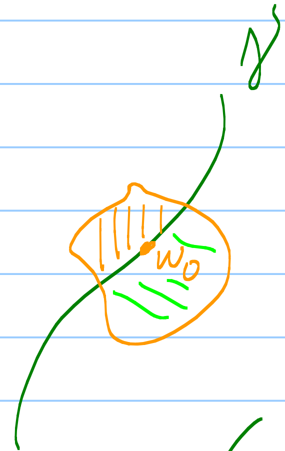
EX: For $\mathbb{R}$: $R(z) = \bar{z}$ ← antiholorphic (conjugate is holo)

R maps one side of $\mathbb{R}$ to other side and fixes $\mathbb{R}$.

<u>For $C_1(0)$</u> : $R(z) = 1/\bar{z}$

<u>For any real analytic γ</u> :

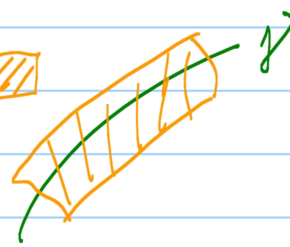




$$R(z) = z\left(\overline{z^{-1}(w)}\right)$$

Famous function: $S(z) = \overline{R(z)}$ is <u>the</u> Schwarz fcn.

It is analytic on ▨



$S(z) = \bar{z}$ on γ.

For $\mathbb{R}$: $S(z) = z$

For $C_1(0)$: $S(z) = 1/z$