Lecture 27 Rouché's theorem, Hurwicz thm

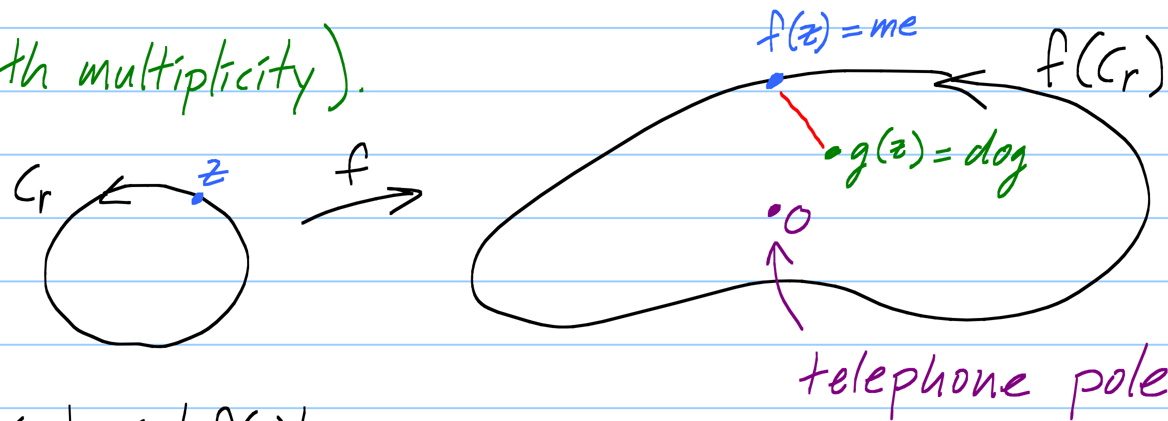
Midterm exam a take-home exm
Out Thurs 2:00pm. Due Tues, 11:59pm
in Gradescope.

Rouché's thm for a disc (or "toy region"): Suppose $f(z)$ and $g(z)$ are analytic on $D_R(a)$ and $0<r<R$. If

$$|f(z)-g(z)| < |f(z)| \text{ on } C_r(a),$$

then f and g have the same number of zeroes inside $C_r(a)$ (counted with multiplicity).

Meaning:



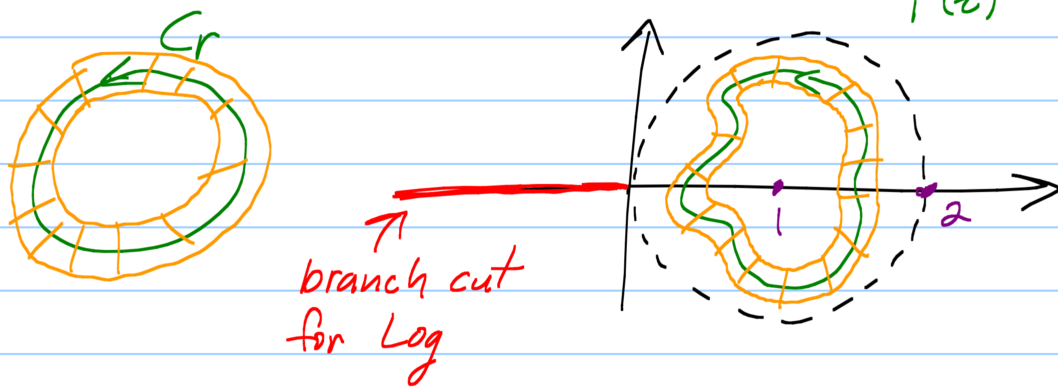
$$\underbrace{|f(z)-g(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{dist me to pole}}$$

Arg Princ: # zeroes f inside C_r = # times $f(z)$ revolves around origin

= same # times dog goes around pole.

Note: Ineq $\Rightarrow$ f nonvanishing on C_r
$\Rightarrow$ g same

Pf: Divide ineq by $|f|$: $\quad \Big|1 - \underbrace{\frac{g(z)}{f(z)}}_{F(z)}\Big| < 1$ on C_r



$$\int_{C_r} \frac{F'}{F}\,dz = \int_{C_r} \frac{d}{dz} \operatorname{Log}(F(z))\,dz = 0$$

$$\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f} \quad \longleftarrow \text{"logarithm diff'tion"}$$

$$\text{So } 0 = \frac{1}{2\pi i}\int_{C_r} \frac{F'}{F}\,dz = \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{g'}{g}\,dz}_{\#\text{zeroes } g} - \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f'}{f}\,dz}_{\#\text{ zeroes of } f}$$

Remark: Same proof extends to "toy regions" and allows zeroes and poles inside:

$$[\#\text{zeroes} - \#\text{poles}] \text{ of } f = \text{same \# for } g.$$

Hurwicz thm: Suppose f_n is a seq of analytic fcns on a domain Ω $\quad f_n \twoheadrightarrow f$. $\quad$ ($\twoheadrightarrow$: unif on compacts)

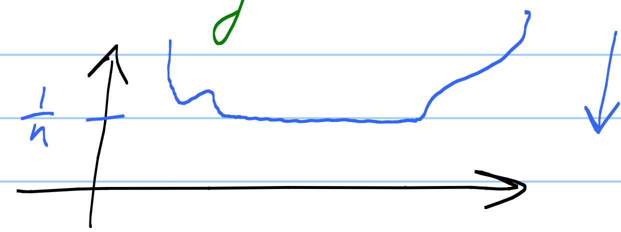
[know f is analytic too.]

If all f_n's are nonvanishing on Ω, then

1) $f \equiv 0$ on Ω, or

2) f nonvanishing too.

Way false $\mathbb{R} \to \mathbb{R}$



$\underline{EX}$: $f_n(z) = z^n \longrightarrow\!\!\!\!\to \equiv 0$ on $D_1(0)$.

$\underline{Pf}$ of Hurwicz: Suppose $f(z_0) = 0$. If $f \not\equiv 0$, then z_0 is an $\underline{\text{isolated}}$ zero. So $\exists\ \overline{D_\varepsilon(z_0)} \subset \Omega$ such that z_0 is the only zero of f in $\overline{D_\varepsilon(z_0)}$.

Let $m = \min\{|f(z)| ;\ z \in C_\varepsilon(z_0)\} > 0$ $|f(z(t))|$ on a compact $[0, 2\pi]$.

$$\underbrace{|f_n(z) - f(z)| < m}_{\text{Get this for } n > N \text{ on compact } C_\varepsilon(z_0) \text{ by } \longrightarrow\!\!\!\!\to} \leq |f(z)| \quad \text{on } C_\varepsilon(z_0)$$

Aha! f_n and f have same # zeroes in $C_\varepsilon(z_0)$, $n > N$.

f_n: none! f: at least one. $\lightning$.

Conclude that f must have been $\equiv 0$.

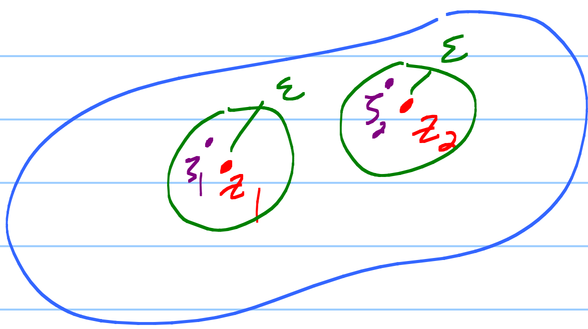
Corollary: Hurwicz #2. $f_n \longrightarrow\!\!\!\!\rightarrow f$

If each f_n is one-to-one, then

1) $f \equiv$ const, or
2) f is one-to-one too.

Pf: Redo pf of Hurwicz #1. Suppose $f(z_1) = f(z_2) = w_0$

If f not const, zeroes of $f(z) - w_0$ would be isolated.



Rouché argument $\Rightarrow$ for $n > \text{Max}(N_1, N_2)$,

$$\begin{pmatrix} \#\text{ zeroes of } f_n(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix} = \underbrace{\begin{pmatrix} \#\text{ zeroes of } f(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix}}_{\geq 1}$$

So $\exists\, \zeta_j \in C_\varepsilon(z_j)$ such that $f_n(\zeta_j) = w_0$, $j = 1, 2$, violating f_n 1-1. ↯

Fun thing: Rouché's $\Rightarrow$ F. Thm. Alg

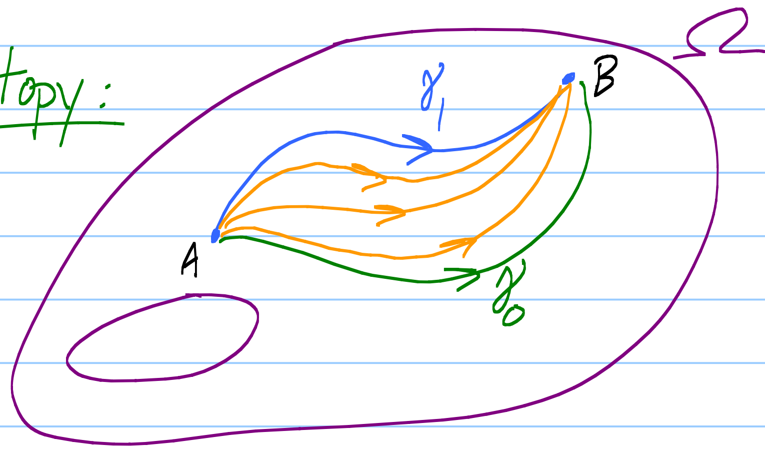
$$f(z) = a_N z^N \qquad g(z) = a_N z^N + \cdots + a_0$$

EX: How many zeroes does $\underbrace{z^5 + 3z^4 + \underbrace{8z^3}_{f(z) \text{ big one}} + z^2 + z + 1}_{g(z)}$ in unit disc?

$$|f(z)-g(z)| = |z^5+3z^4+z^2+z+1| \le \underbrace{1+3+1+1+1}_{7} \le |8z^3| \quad \text{on } |z|=1.$$

$g(z)$ has 3 zeroes in $D_1(0)$.

Homotopy:



$\gamma_j : z_j(t),\ 0 \le t \le 1$

$\gamma_0 \sim \gamma_1$: "γ_0 homotopic to γ_1 in Ω" means that there is a continuous $H : [0,1]\times[0,1] \to \Omega$ such that

$$\begin{cases} H(0,t) = z_0(t) & 0 \le t \le 1 \\ H(1,t) = z_1(t) & 0 \le t \le 1 \end{cases}$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$$

Think: $\gamma_s : z_s(t) = H(s,t) \leftarrow$ orange curves.