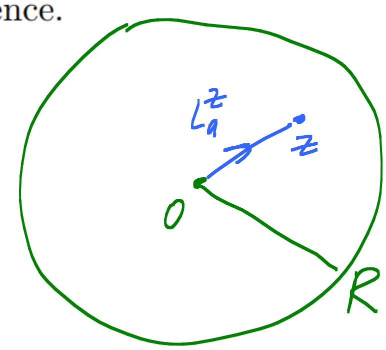
Practice problems, review

2. Prove that power series can be integrated term by term. To be precise, suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges on the disc $D_R(0)$ to an analytic function $f(z)$. Prove that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ also has radius of convergence R and that this series converges to an analytic anti-derivative of $f(z)$ inside the circle of convergence.



$$F(z) = \int_{L_a^z} f(w)\,dw$$

$$= \sum_0^\infty a_n \underbrace{\int_{L_a^z} w^n\,dw}_{\frac{1}{n+1} z^{n+1}}$$

$R_F \geq R_f$

$R_F > R_f \implies$ power series for f converges on disc bigger than R_f.

Fact: Can diff and integrate Laurent expansions in same way, but the residue term gets in the way when you try to integrate.

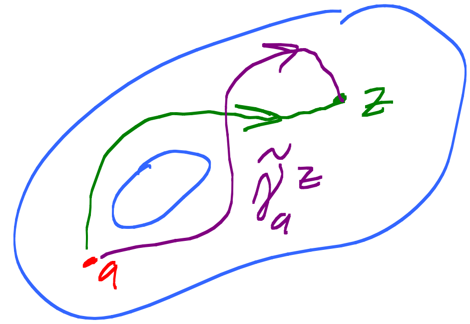
Lemma: f has an analytic antiderivative on a domain $\Omega \iff \int_\gamma f\,dz = 0$ for every closed contour γ in Ω.

Pf: $\implies$ $f = F'$. Then $\int_\gamma f\,dz = \int_\gamma F'\,dz = 0$, γ closed.

$\impliedby$ Suppose $\int_\gamma f\,dz = 0$, γ closed.

Try to define

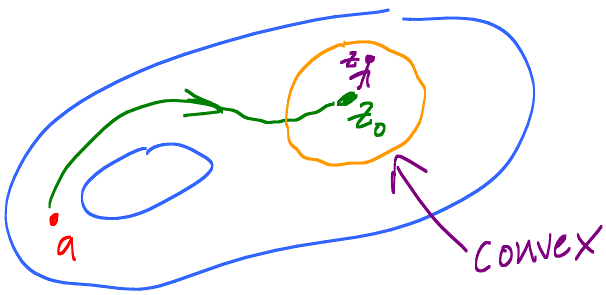
$$F(z) = \int_{\gamma_a^z} f(w)\,dw.$$



<u>Claim</u>: F well defined. ✓

$\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is closed.

<u>Trick</u>:



$$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$$

$$\frac{d}{dz} F(z) = \frac{d}{dz}\Big(\underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{\text{const}} + \int_{L_{z_0}^z} f\,dw \Big) = f(z)$$

<u>Fact</u>: Suppose f analytic on $D_r(a) - \{a\} =: \Omega$

f has an analytic antiderivative on Ω

$\iff \operatorname{Res}_a f = 0.$

<u>Pf</u>: $\int_{\gamma_a^z} \sum = \sum \int$

$$a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(w)\,dw$$

3. Suppose that f and g are analytic in a neighborhood of a. If f has a simple zero at a, then

$$\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

Prove a similar formula in case f has a double zero at a, i.e., in case f is such that $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$.

$$f(z) = \underbrace{a_0}_{=0} + \underbrace{a_1}_{=0}(z-a) + \underbrace{a_2}_{= \frac{f''(a)}{2!}}(z-a)^2 + \underbrace{a_3}_{= \frac{f'''(a)}{3!}}(z-a)^3 + \cdots$$

$$= (z-a)^2 \Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{F(z)} \Big]$$

Note: $a_2 = F(a) = \frac{f''(a)}{2!}$, $\quad a_3 = \frac{F'(a)}{1!} = \frac{f'''(a)}{3!}$

$$\frac{g(z)}{f(z)} = \frac{1}{(z-a)^2} \Bigg[\underbrace{\frac{g(z)}{F(z)}}_{A_0 + A_1(z-a) + \cdots} \Bigg] \leftarrow \text{analytic near } a.$$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \text{power series}$$

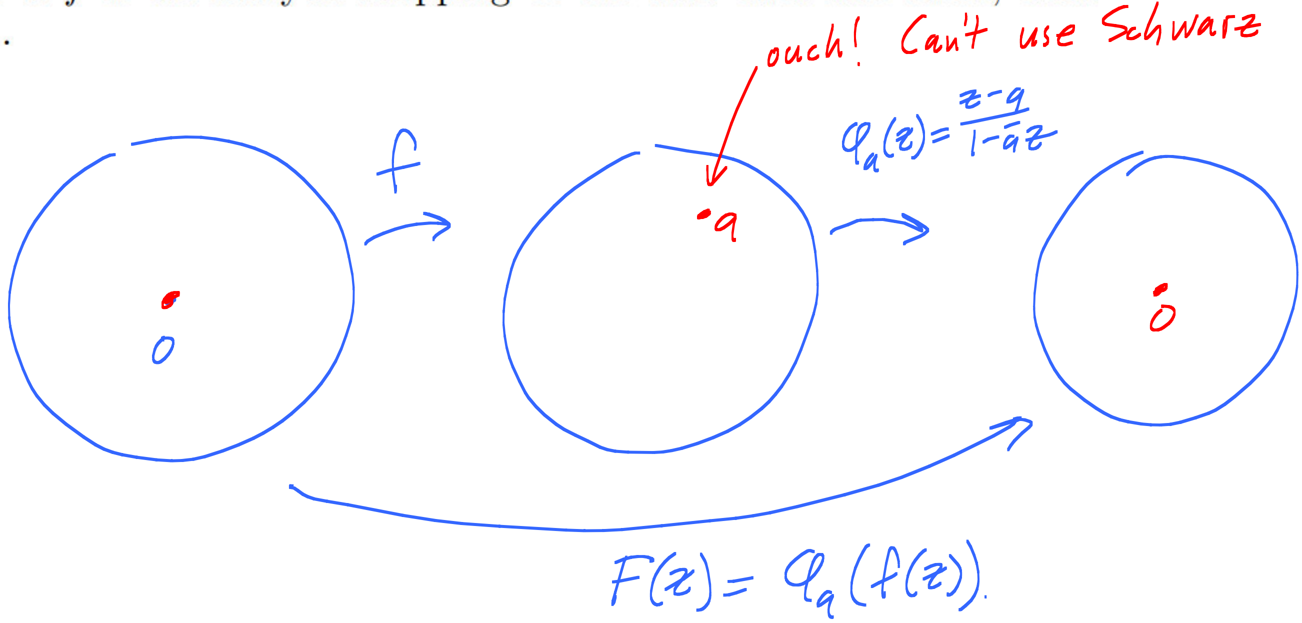
$$\mathrm{Res}_a \frac{g}{f} = A_1 = \frac{\frac{d}{dz}\left[\frac{g}{F}\right]\Big|_{z=a}}{1!} = \frac{g'(a)F(a) - g(a)F'(a)}{F(a)^2}$$

Hated formula:

$$\mathrm{Res}_a \frac{g}{f} = \lim_{z \to a} \frac{\frac{d}{dz}\left[(z-a)^2 \frac{g(z)}{f(z)}\right]}{1!}$$

$$= \frac{g'(a)\frac{f''(a)}{2!} - g(a)\frac{f'''(a)}{3!}}{\left[\frac{f''(a)}{2!}\right]^2}$$

5. Show that if f is an analytic mapping of the unit disk into itself, then $|f'(0)| \le 1$.



Schwarz: $|F'(0)| \le 1$

$$F'(z) = \varphi_a'(f(z))\, f'(z)$$

$$\underbrace{|F'(0)|}_{\le 1} = |\underbrace{\varphi_a'(a)}_{\frac{1}{1-|a|^2}}\, f'(0)| \le 1$$

Conclude $|f'(0)| \le 1-|a|^2$

< 1 if

$a \ne 0$.

Schwarz $\rightarrow$ (≤ 1 if $a=0$.)

What if $|f'(0)| = 1$? Then $a=0$. Schwarz part 2

$\Rightarrow$ $f(z) = \lambda z$, $|\lambda| = 1$.

6. Suppose that f is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if f has a zero of order n at the origin, then

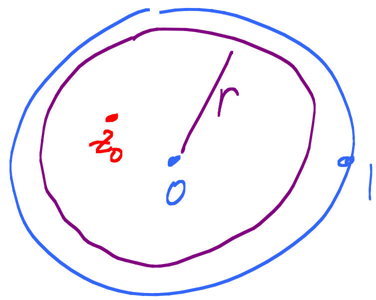
$$|f(z)| \leq |z|^n$$

for $|z| < 1$. How big can $|f^{(n)}(0)|$ be?

Know $f(z) = z^n \underbrace{F(z)}_{[a_n + a_{n+1} z + \cdots]}$, $F(0) \neq 0$ and analytic

$F(0) = a_n = \dfrac{f^{(n)}(0)}{n!}$

$F(z) = \dfrac{f(z)}{z^n}$



$|z_0| < r < 1$

$$|F(z_0)| \leq \max_{C_r} |F| = \max_{C_r} \frac{|f(z)|}{r^n} \leq \frac{1}{r^n} \searrow 1 \text{ as } r \nearrow 1,$$

Get $|F(z_0)| \leq 1$; $|f(z_0)| \leq |z_0|^n$ ✓

Also get $\Big|\underbrace{F(0)}_{\frac{f^{(n)}(0)}{n!}}\Big| \leq 1$. So $|f^{(n)}(0)| \leq n!$

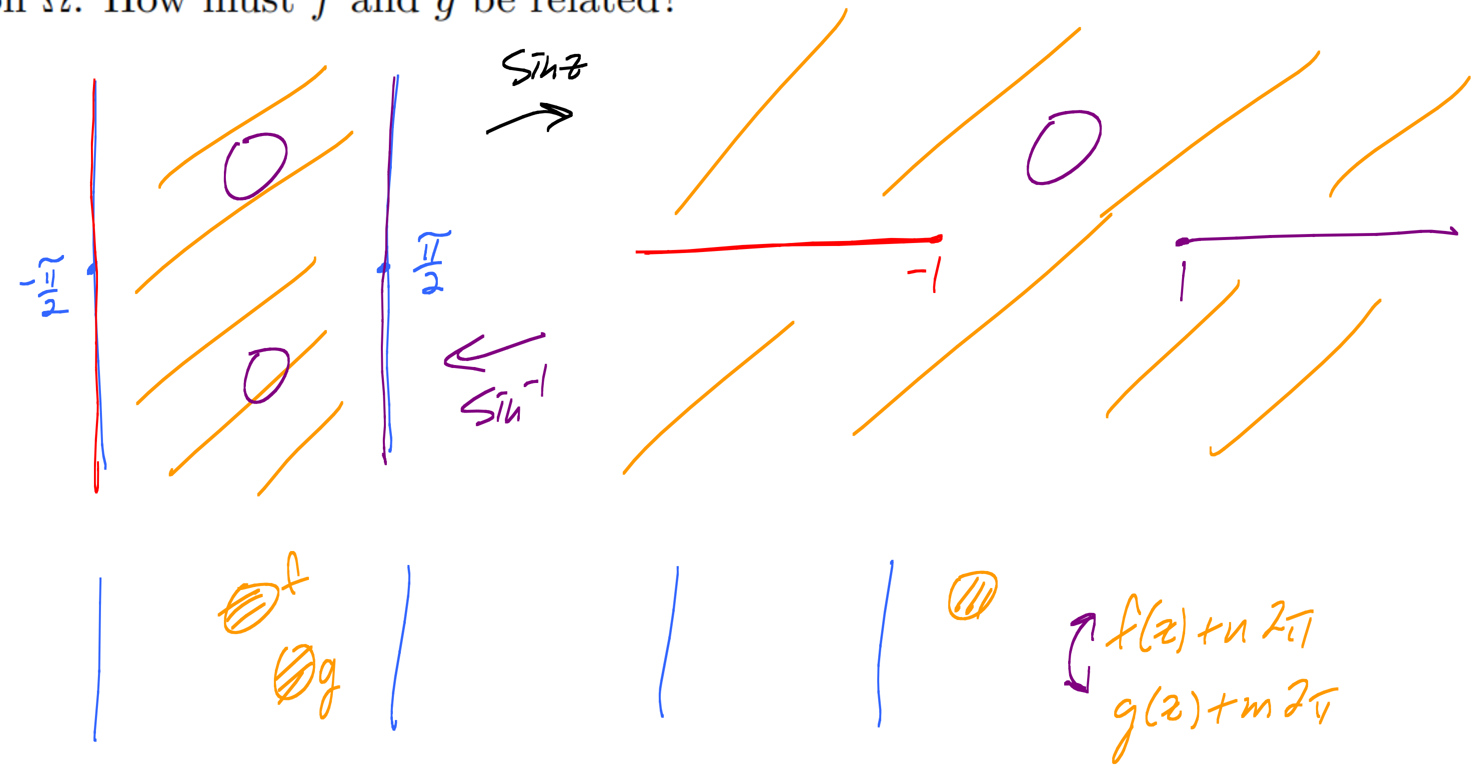
equality? $f(z) = \eta z^n$

8. Suppose f and g are analytic functions on a domain Ω and that

$f(z)+2n\pi$ open mapping thm

$$\sin(f(z)) \equiv \sin(g(z))$$

on Ω. How must f and g be related?



7. $|f(z)| \le c(1+|z|^N)$ for $z \in \mathbb{C}$. f entire.

Cauchy est: $$|f^n(0)| \le \frac{n! \max_{C_R(0)} |f|}{R^n} \le \frac{n!\,(1+R^N)}{R^n}$$

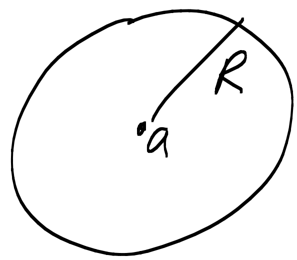
Aha! If $n > R$, see $|f^{(n)}(0)| \to 0$

as we let $R \to \infty$ in Cauchy est.

Power series: $f(z) = a_0 + a_1 z + \cdots + a_N z^N + 0 + \cdots$

might be zero.

<u>Or</u>:



$f^{(n)}(a) \equiv 0$

when $n = N+1$

$f^{(N+1)}(z) \equiv 0 \implies f$ poly deg N <u>or less</u>