Lecture 29 Consequences of the Cauchy Theorem

Midterm exam due in GS 11:59 pm Tues

Cauchy theorem Suppose f is analytic on a simply connected domain Ω. Then

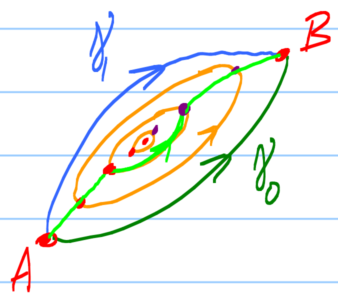
1) $\int_\gamma f \, dz = 0$, γ in Ω closed

2) $\int_{\gamma_A^B} f \, dz = \int_{\tilde{\gamma}_A^B} f \, dz \leftarrow \int_A^B f \, dz$

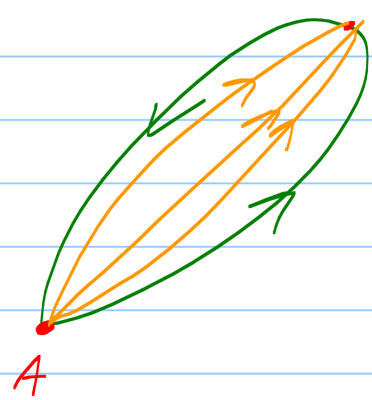
indep of path

Remark Ω simply connected $\Longleftrightarrow$ for any $A, B \in \Omega$ $\gamma_A^B \sim \tilde{\gamma}_A^B$.

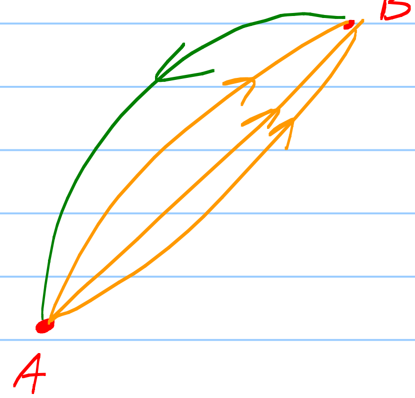
$(\Rightarrow)$



$(\Leftarrow)$

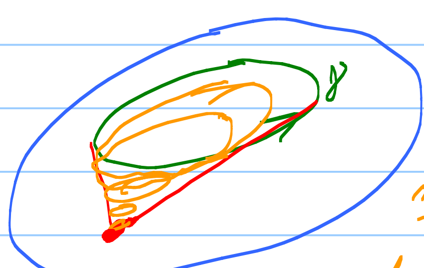


$\gamma = \gamma_0 \cup (-\gamma_1)$



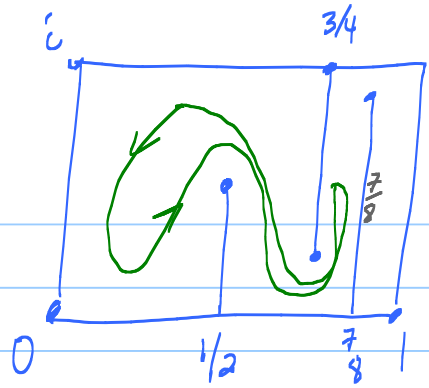
Examples of s.c. domains:

1) Convex domains are s.c.



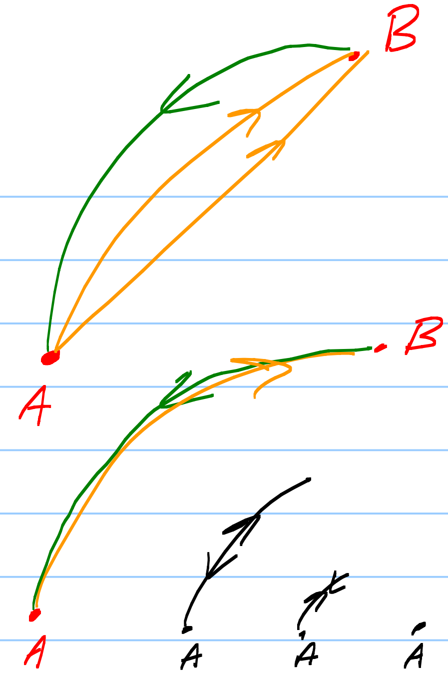
"tornado" down to pt.

2)



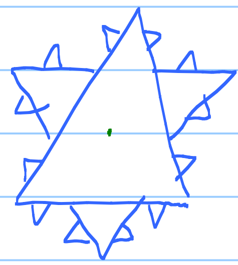
etc.

Spaghetti again.



Suck it in like spaghetti.

3) Serpinski snowflake



etc.

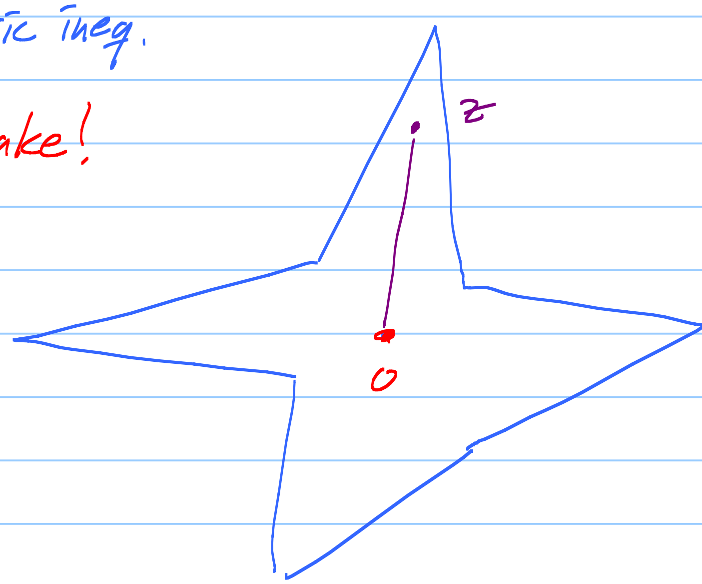
$A \leq \frac{L^2}{4\pi}$ Isoperimetric ineq.

bounded

$\to \infty$ in snowflake!

3) Starshaped domains

tornado method



Thm: Suppose f is analytic on a s.c. domain Ω.

A) $\exists$ analytic antiderivative F on Ω with $F' = f$.

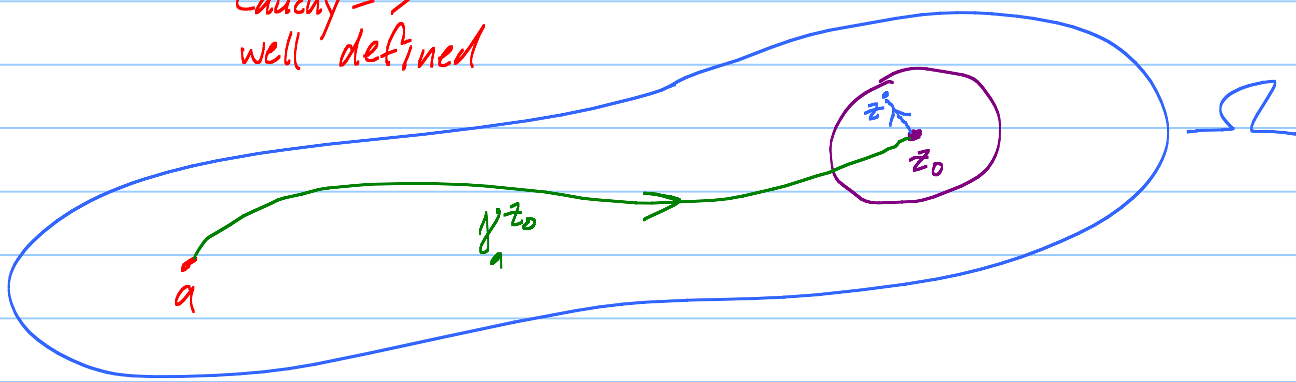
B) If f is nonvanishing on Ω, then $\exists$ analytic G on Ω with $f = e^G$. $\left[\frac{d}{dz} G = \frac{f'}{f} \right]$

And $\exists$ analytic H with $f = H^N$ $\quad \left[H = e^{G/N} \right]$

c) If u is harmonic on Ω, then u has a harmonic conjugate on Ω.

<u>Pf</u>: A) "<u>Define</u>" $\quad F(z) = \int_{\gamma_a^z} f(w)\,dw$

Cauchy $\Rightarrow$ well defined



$$F(z) = \underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{\text{const.}} + \underbrace{\int_{L_{z_0}^{z}} f\,dw}_{\text{anti-deriv for } f \text{ on convex disc.}}$$

So $F'(z) = f(z)$ on disc. $\leftarrow$ arb z_0

B)
$$g(z) = \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw$$
$\leftarrow$ nonvanishing!

Get $\quad g'(z) = \dfrac{f'}{f}$

Trick: $\quad \dfrac{d}{dz}\left(\dfrac{e^g}{f}\right) \equiv 0$

So $\dfrac{e^g}{f} = \text{const.}$

$$f = ke^g = e^{\text{Log}K} e^g = e^{\overbrace{(\text{Log}K + g)}^{G}}$$

c) u harmonic. C-R Equs $\Rightarrow$ $F = u_x - iu_y$ analytic.

Get $\tilde{f}$ antideriv of $F = u_x - iu_y$.

Think: $f = u + iv$ ← analytic

$f' = \underbrace{u_x + iv_x}_{\text{red box \#1}} = u_x - iu_y$

CR Equs.

Now $F = \tilde{f}'$

$u_x - iu_y = \tilde{u}_x - i\tilde{u}_y$ ← $\nabla u \equiv \nabla \tilde{u}$

So $u = \tilde{u} + c$ on Ω.

Aha! $\tilde{f} = \tilde{u} + i\tilde{v} = (u - c) + i\tilde{v}$ is analytic.

So is $u + i\tilde{v}$

harm conj

Thm: f analytic on a domain has a complex antiderivative $\Longleftrightarrow$ $\int_\gamma f\,dz = 0$ for all closed γ.

Pf: In review lecture.

Cor: f analytic on $D_r(z_0) - \{z_0\}$. [Isolated sing.]

f has an analytic antideriv on $D_r(z_0) - \{z_0\}$

$\iff \operatorname{Res}_{z_0} f = 0.$

Pf: $(\Rightarrow)$ $a_{-1} = \frac{1}{2\pi i}\int_{C_\rho(z_0)} \overbrace{f(w)}^{= F'(w)}\, dw = 0$

$(\Leftarrow)$ Laurent exp converge unif on $\underbrace{\{\varepsilon < |z-z_0| < \rho < r\}}_{A_\varepsilon^\rho}$

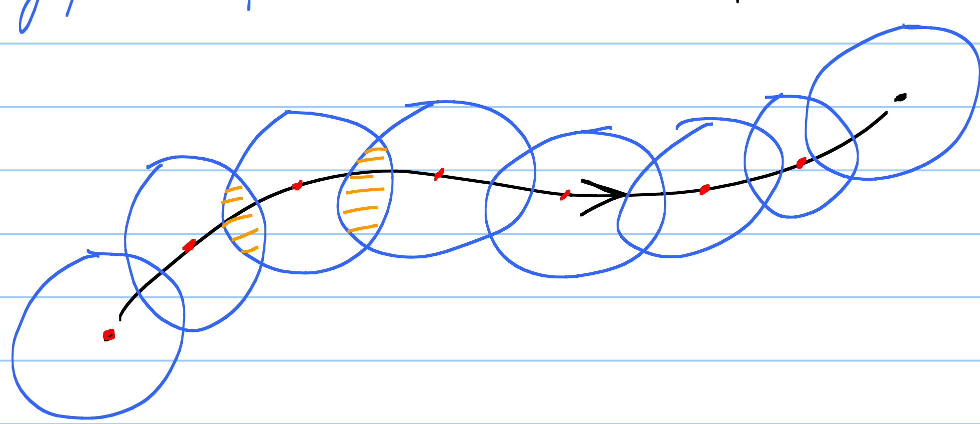
Given closed γ in $D_r(z_0) - \{z_0\}$, $\exists$ A_ε^ρ containing γ.

$$\int_\gamma f\, dw = \int_\gamma \sum_{-\infty}^{\infty} a_n (w-z_0)^n\, dw$$

$$= \sum_{-\infty}^{\infty} a_n \int_\gamma \underbrace{(w-z_0)^n}_{= \text{deriv except } n=-1 \text{ case}}\, dw = \underbrace{a_{-1}}_{=0} \int_\gamma \frac{1}{w-z_0}\, dw$$

✓

Hungry caterpillar idea γ simple curve.



Simply connected caterpillar

Easy to see $\int_{\gamma_A^B} f\, dz$ well defined in worm. Use when γ_A^B is just cont.