Lecture 35 More about the Dirichlet problem

HWK 7 due tomorrow Tues in GS.

Fun things to do: poly $p(x,y) = p\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right) = P(z,\bar z)$

$$= \sum_{n,m=0}^{N} c_{nm} \underbrace{z^n \bar z^m}$$

$= 1 \quad n=m$

$= z^{n-m} \quad n>m$

$= \bar z^{m-n} \quad m>n$

on $|z|=1$

$$= c_{00} + p(z) + \overline{q(z)} \quad \text{on } C_1(0), \quad p, q \text{ polys!}$$

Aha! $p(z)$, $q(z)$ are analytic. So real and imag parts are harmonic: Write $c_{00} + p(z) + \overline{q(z)}$

$$= u(x,y) + i\,v(x,y)$$

real harmonic polys!

Hmmm: on $C_1(0)$: $p(x,y) = u(x,y) + i\,\underbrace{v(x,y)}_{\equiv 0}$

real

Fact: On discs, poly boundary data yield poly sol^n's to D. Prob.

Something about Poisson and Dirichlet (see home page).

Fun: Algebra of $p(x,y)$ on $C_1(0)$ separates points.

Stone-Weirstraß $\Rightarrow$ polys dense among continuous fcns on $C_1(0)$.

Idea: Given cont. real valued fcn φ on $C_1(0)$,

take seq polys $p_n(x,y) \rightrightarrows \varphi$ on $C_1(0)$.

Solve D. Prob for $p_n(x,y)$. Get $\wp_n$'s.

Show $\wp_n \rightrightarrows$ sol^n to D. Prob for φ on $\overline{D_1(0)}$.

Important lemma: If a seq of harmonic fcns u_n converges unif on compact subsets of a domain Ω to u, then u is harmonic.

Pf: MA 504 implies that u is continuous.

Unif conv on circles $\Rightarrow$

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} u_n(z_0+re^{i\theta})\,d\theta}_{u_n(z_0)} \longrightarrow \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta = u(z_0)$$

So u satisfies ave prop $\Rightarrow$ u harmonic.

Back to D. Prob. $\wp_n \longrightarrow$ harmonic fcn.

Hmmm. How to use Max princ?

Like a HWK prob. $p_n \rightrightarrows \varphi$ on $C_1(0)$.

So p_n unif Cauchy on $C_1(0)$.

Max princ $\Rightarrow$ p_n are unif Cauchy on $\overline{D_1(0)}$.

So $p_n \rightrightarrows$ unif on $\overline{D_1(0)}$ to Φ.

Φ is cont on $\overline{D_1(0)}$, $\Phi = \varphi$ on $C_1(0)$,

and Φ harm on $D_1(0)$. Φ solves D. Prob.

Interesting fact Ellipses have poly $\rightarrow$ poly D. Prob property.

Pf: Ellipse: $\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}_{\rho(x,y)} = 0$

$\rho(x,y)$ "defining fcn" for E.

$E = \{(x,y) : \rho(x,y) < 0\}$.

ρ is a degree two poly in x and y.

P_N = Vector space of polys of x and y of deg N or less.

$\tilde{\nabla}$: Fisher map: poly $q(x,y) \longmapsto \Delta(\rho \cdot q)$

increases deg by 2

decreases deg by 2!

So $\tilde{\nabla} : P_N \to P_N$ linear.

<u>Claim</u>: $\tilde{\varphi}$ is 1-1.

<u>Why</u>: If $\tilde{\varphi}(q) = 0$

$$\Delta(\rho q) \equiv 0$$

So ρq is <u>harmonic</u>. But $\rho \equiv 0$ on bndry!

Max princ $\Rightarrow \rho q \equiv 0. \Rightarrow q \equiv 0.$

<u>Claim</u> $\tilde{\varphi}$ is onto. Linear alg fact:

$$\tilde{\varphi} : P_N \xrightarrow{1-1} P_N$$ ← finite dim v. space

$$\Rightarrow \tilde{\varphi} \text{ onto.}$$

Aha! Can use $\tilde{\varphi}$ to solve D. Prob on E with poly data.

Given poly data q. Let Δq.

$\tilde{\varphi}$ onto $\Rightarrow \exists\, p$ with $\tilde{\varphi} p = \Delta q$.

$$\Delta(\rho p) = \Delta q$$

Aha! $q - \rho p$ solves D. Prob with bndry data q.

$q - \rho p = q$ on bndry ✓

$$\Delta(\underbrace{q - \rho p}_{\text{harmonic ✓}}) = \Delta q - \Delta(\rho p) \equiv 0.$$

Khavinson–Shapiro conjecture: Only discs and ellipses have poly → poly D. Prob. property.

What about rational data?

On unit disc: $r(x,y) = R(z,\bar{z}) = R(z,\frac{1}{z})$ on $C_1(0)$.

$S(z) = 1/z$ is the Schwarz fcn for unit disc.

$\frac{1}{z} = \bar{z}$ on $C_1(0)$.

So $R(z,\bar{z}) = R(z,\frac{1}{z}) \underset{\substack{\uparrow \\ \text{partial fractions}}}{=} \sum \underbrace{\frac{A_{kn}}{(z-a_n)^k}}_{a_n \in D_1(0)} + \sum \underbrace{\frac{B_{kn}}{(z-b_n)^k}}_{b_n \text{ outside } C_1(0)}$

$\underset{\substack{\uparrow \\ \text{on } C_1(0)}}{=} \underbrace{\sum \frac{A_{kn}}{(\frac{1}{\bar{z}}-a_n)}}_{\substack{\text{singularities} \\ \text{at } 1/\bar{a}_n \\ \text{outside } C_1(0)}} + \sum \frac{B_{kn}}{(z-b_n)^k}$

$= \underbrace{\overline{Q(z)} + r(z)}_{\text{solve D. Prob.}}$

Fact: On discs, rational → rational for D. Prob.

B., Ebenfeld, Khavinson, Shapiro: Discs are only domains with this feature!

Important formula

$$P(a,\theta) = \frac{1}{2\pi}\frac{1-|a|^2}{|a-e^{i\theta}|} = \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{a+e^{i\theta}}{a-e^{i\theta}}\right]$$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\,u(e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\,\underbrace{\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,u(e^{i\theta})\,d\theta}_{\text{analytic in } z}$$

Harmonic conjugate v for u = Im part of $\int$.

Hilbert transform: singular integral operator