Lecture 38 The General residue theorem

<u>General Cauchy thm</u> Ω domain, Γ cycle, f analytic on Ω.

If $\Gamma \sim 0$ in Ω, then

1) $\int_\Gamma f \, dz = 0$

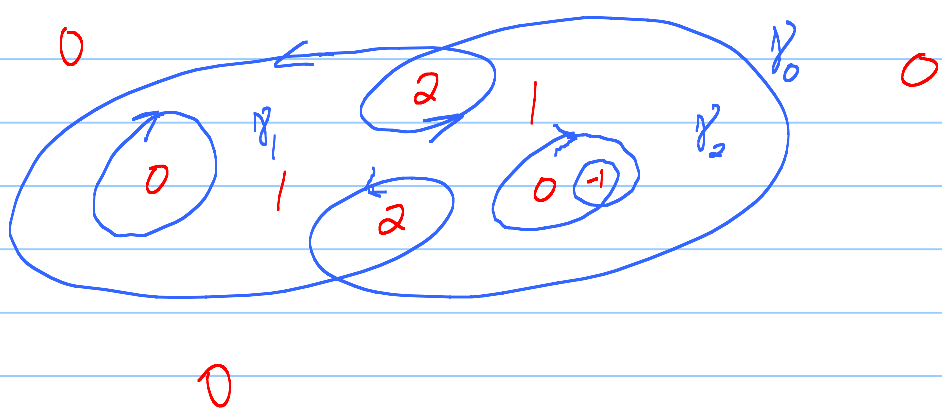
$a \notin tr(\Gamma)$, 2) $f(a) \, Ind_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz$

<u>Def^{n}s</u>: $\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$, γ_j closed curves in Ω.

$$Ind_\Gamma(a) = \frac{1}{2\pi i} \sum_1^n \int_{\gamma_j} \frac{1}{z-a} dz$$

$$\int_\Gamma f \, dz = \sum_1^n \int_{\gamma_j} f \, dz$$

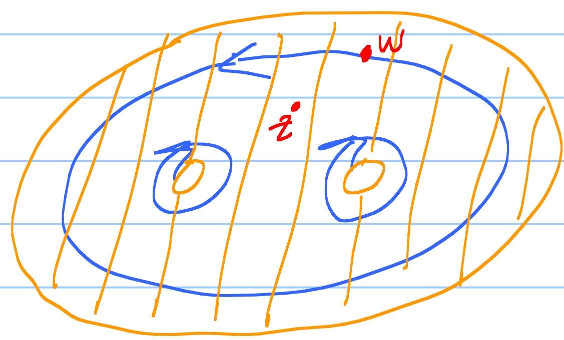
$\Gamma \sim_\Omega 0$: $Ind_\Gamma(a) = 0$ for $a \in \mathbb{C} - \Omega$.



<u>Fact</u>: Set $K = tr(\Gamma) \cup \{z : Ind_\Gamma(z) \neq 0\}$ is a compact set in Ω. [K is closed ✓ bounded ✓]

<u>Pf</u>: recap

$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma);\ Ind_\Gamma(z) = 0 \}$



Ω

Facts: $\tilde{\Omega}$ open $\subset \mathbb{C} - \Omega$. $\quad \mathbb{C} = \Omega \cup \tilde{\Omega}$.

$\tilde{\Omega}$ contains the unbounded component of $\mathbb{C} - tr(\Gamma)$.

So $\exists\ R > 0$ such that $\{ z : |z| > R \} \subset \tilde{\Omega}$.

DQ

$\int_\Gamma g(z,w)\,dw$

$$H(z) = \begin{cases} f(z) \int_\Gamma \frac{dw}{z-w} - \int_\Gamma \frac{f(w)}{z-w}\,dw & z \in \Omega \quad z \notin tr(\Gamma) \\ -\int_\Gamma \frac{f(w)}{z-w}\,dw & z \in \tilde{\Omega} \end{cases}$$

Last time: H is bnd and entire! $\to 0$ at ∞.

Liouvilles $\Rightarrow H \equiv 0$.

Top line of # def^n is the Cauchy Integral Formula (CIF)!

Fun fact: CIF $\Rightarrow$ Cauchy Thm.

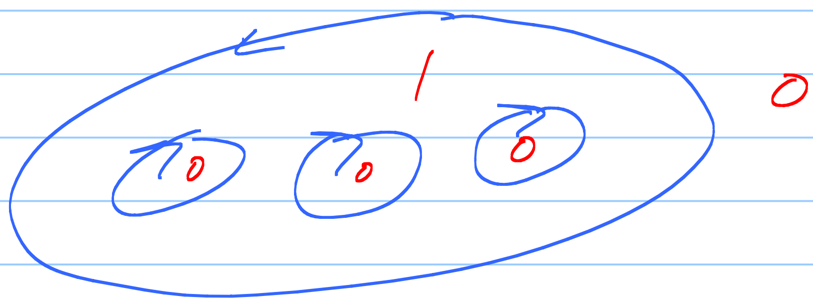
Why: Pick $z_0 \in \Omega - tr(\Gamma)$.

Define $F(z) = (z - z_0) f(z)$

CIF for F at z_0:

$$0 = F(z_0)\,\mathrm{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)(z-z_0)}{\underbrace{z-z_0}_{f(z)}}\,dz$$

Word A cycle is called simple if $\mathrm{Ind}_\Gamma(a)$ is only equal to zero or one on $\mathbb{C} - tr(\Gamma)$.



Think: $\{z : \mathrm{Ind}_\Gamma(z) = 1\}$ is the "inside" of Γ

$\{z : \mathrm{Ind}_\Gamma(z) = 0\}$ is the "outside"

"If f is analytic "inside" and on a simple cycle Γ, then $\int_\Gamma f\,dz = 0$."

Remark: Terminology: f "analytic on a compact K"

means $\exists$ open Ω with $K \subset \Omega$, analytic F on Ω such that $f = F$ on K.

New improved Cauchy thm for simp conn. domains:

f analytic on s.c. domain Ω, γ closed curve in Ω,

1) $\int_\gamma f\,dz = 0 \leftarrow$ not new. $\gamma \sim \{z_0\}$.
homotopic

2) $f(z_0)\,\mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z_0}\,dw$

Pf: $\gamma \sim 0$ because $\int_\gamma \frac{1}{z-a}\,dz = 0$ by Cauchy thm when $a \in \mathbb{C} - \Omega$.
homologous to zero

General residue thm: Ω is a domain, Γ a cycle in Ω, $\Gamma \sim 0$ in Ω, f analytic on Ω except at finitely many isolated singular pts $A = \{a_1, a_2, \ldots, a_N\} \leftarrow$ distinct in $\Omega - \mathrm{tr}(\Gamma)$.

Then
$$\int_\Gamma f\,dz = 2\pi i \sum_{n=1}^{N} \mathrm{Ind}_\Gamma(a_n)\,\mathrm{Res}_{a_n} f$$

[Simple cycle: $\int_\Gamma f\,dz = 2\pi i \left(\sum \text{Residues of } f \text{ "inside" } \Gamma \right)$]

Pf: Let $\tilde{\Omega} = \Omega - A$

Let $n_j = \mathrm{Ind}_\Gamma(a_j)$.

Def^n: $n_j C_\varepsilon(a_j)$: $z(t) = a_j + \varepsilon e^{in_j t}$ $0 \le t \le 2\pi$

"Go around $C_\varepsilon(a_j)$ n_j times"

Easy facts $\mathrm{Ind}_{n_j C_\varepsilon(a_j)}(a_j) = n_j$.

$$\int_{n_j C_\varepsilon(a_j)} f \, dz = n_j \underbrace{\int_{C_\varepsilon(a_j)} f \, dz}_{= 2\pi i \, \mathrm{Res}_{a_j} f}$$

Can take $\varepsilon > 0$ so small that

1) Each $\overline{D_\varepsilon(a_j)} \subset \Omega$

2) $\overline{D_\varepsilon(a_j)} \cap tr(\Gamma) = \emptyset$

3) $\overline{D_\varepsilon(a_j)} \cap \overline{D_\varepsilon(a_k)} = \emptyset$, $j \neq k$.

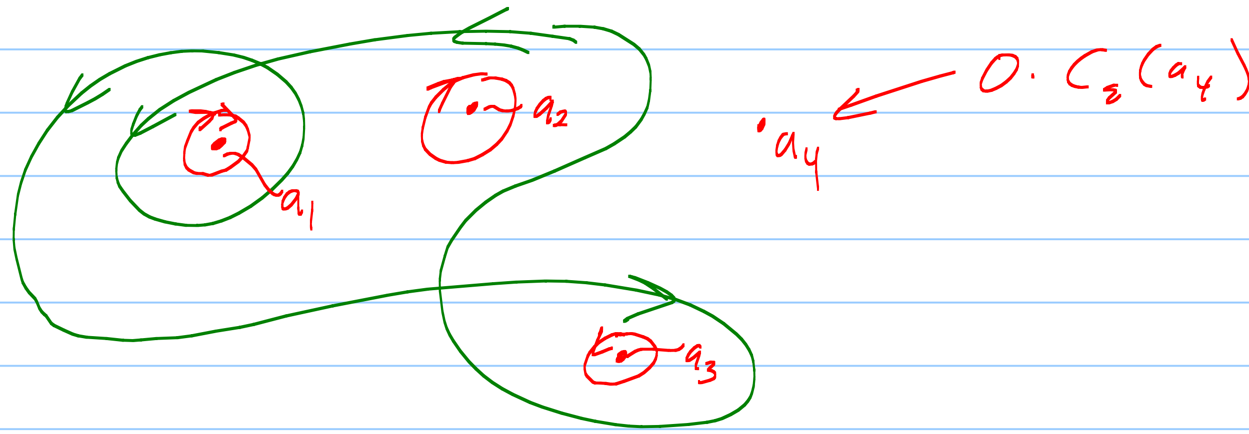
Aha! $\tilde{\Omega} = \Omega - A$

$$\tilde{\Gamma} = \Gamma \cup \left[\bigcup_{j=1}^{N} -\underset{\mathrm{Ind}_\Gamma(a_j)}{n_j} C_\varepsilon(a_j) \right]$$

Note: $\mathrm{Ind}_{\tilde{\Gamma}}(a_j) = \mathrm{Ind}_\Gamma(a_j) + (-n_j) +$ 0's from a_k $k \neq j$.

General Cauchy thm can be applied:

$$0 = \int_{\tilde{\Gamma}} f \, dz = \int_{\Gamma} f \, dz + 2\pi i \sum_{j=1}^{N} (-n_j) \operatorname{Res}_{a_j} f \checkmark$$



Remark: Not necessary to let A be finite, just discrete.

[because $K = tr(\Gamma) \cup \{z : \operatorname{Ind}_\Gamma(z) \neq 0\}$ is compact. If $A \cap K$ is infinite, get limit pt. ↯.]

Even though Σ looks infinite, only finitely many $\neq 0$.