Lecture 40 Mittag-Leffler, Weierstraß, infinite products

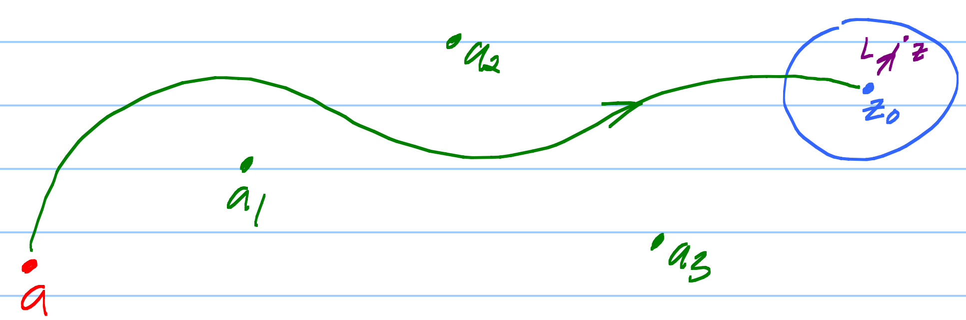
Practice problems for final on home page

Pf that M-L $\Rightarrow$ Weierstraß: $\exists$ entire fcn f with zeroes of multiplicity m_n at $a_n \to \infty$.

M-L $\Rightarrow$ $\exists$ F meromorphic on $\mathbb{C}$ with poles at a_n with principal part $\frac{m_n}{z-a_n}$ at a_n.

Last time showed $\underline{f(z)=\exp\left(\int_{\gamma_a^z} F(w)\,dw\right)}$ well defined via residue thm.

Step 2 f is analytic on $\mathbb{C}-A$

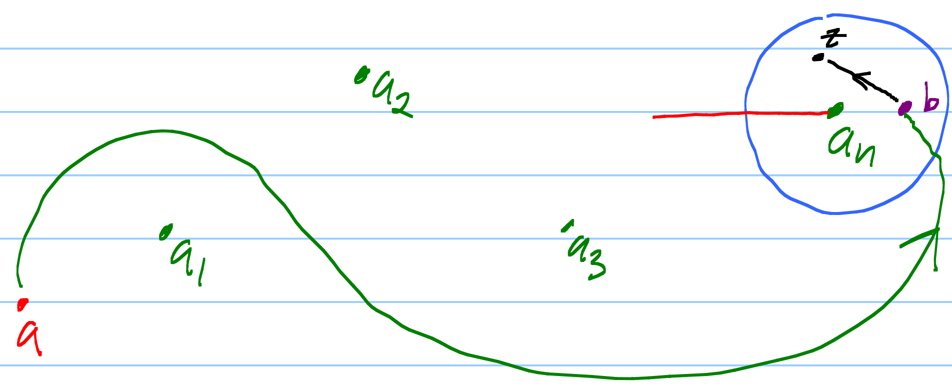


$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^{z_0}} F\,dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^{z}} F\,dw}_{\text{derivative } = F(z)} \right) \checkmark$$

see $f'(z) = \underbrace{\exp\left(\quad\right)}_{f(z)} F(z)$

$$\frac{f'(z)}{f(z)} = F(z) \leftarrow \text{hopefull}$$

<u>Step 3</u> a_n removable sing for f. In fact, a zero of mult m_n.



$$f(z)=\exp\left(\underbrace{\int_{\gamma_a^b} F(w)\,dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{w-a_n}}_{m_n\left[\mathrm{Log}(z-a_n)-\underbrace{\mathrm{Log}(b-a_n)}_{\text{const}}\right]} + \underbrace{H(w)}_{\text{analytic}}\,dw\right)$$

$$f(z)=\exp\left(\text{const} + m_n\,\mathrm{Log}(z-a_n) + \underbrace{h(z)}_{\text{antider of } H}\right)$$

$$=\underbrace{\exp\left(m_n\,\mathrm{Log}(z-a_n)\right)}_{(z-a_n)^{m_n}}\left[\text{Analytic and } \neq 0\right]$$

Aha! Analytic above branch cut, and below and limits agree on cut. Analytic extension across cut! (Morera's).

<u>Cor</u> Combine M-L and Weier to be able to prescribe finitely terms in power series at seq $a_n \to \infty$.

Cor Meromorphic fcns on $\mathbb{C}$ $= \left\{ \frac{f}{g} : f, g \text{ entire}, g \not\equiv 0 \right\}$.

Pf: Suppose F mero on $\mathbb{C}$ with poles of order m_n at discrete $a_n \to \infty$. Weir $\Rightarrow \exists$ entire g with zeroes of mult m_n at a_n. $[g \not\equiv 0]$.

Aha! $gF := f$ has removable sing at a_n's. ✓

Infinit products $\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n$

where $p_n \neq 0$, $\forall n$.

Lemma If limit exists, $\neq 0$, then $p_n \to 1$ as $n \to \infty$.

Pf: $$p_N = \frac{\prod_1^N p_n}{\prod_1^{N-1} p_n} \to \frac{L}{L} = 1$$

Traditional to write $\prod_{n=1}^{\infty} (1 + a_n)$ instead.

Lemma $\prod_{n=1}^{\infty} (1 + a_n)$ converges to $L \neq 0$

$$\Longleftrightarrow \sum_{n=1}^{\infty} \operatorname{Log}(1 + a_n) \text{ converges}$$

Pf: $(\Leftarrow)$ easy. $\prod_{n=1}^{N} (1 + a_n) = \exp\left(\sum_1^N \operatorname{Log}(1 + a_n) \right)$

$(\Rightarrow)$ Read Stein p. 141.

Fact $\operatorname{Log}(1+z)$ analytic on $D_1(0)$ and has a simple zero. And $\left.\frac{d}{dz}\operatorname{Log}(1+z)\right|_{z=0} = 1$

So $(1-\varepsilon)|a_n| \le |\operatorname{Log}(1+a_n)| \le (1+\varepsilon)|a_n|$. $(*)$

Given ε with $0<\varepsilon<1$, $\exists\ \delta>0$ such that $(*)$ holds when $|a_n|<\delta$.

Def^n: $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely when

$$\sum_{n=1}^{\infty}|a_n| < \infty.$$

So, absolute conv $\Longrightarrow$ conv.

Fact If $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on compact sets of a domain Ω, f_n analytic, then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely and uniformly on compacts on Ω to analytic limit.

EX: $f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$ $\leftarrow$ entire. Simple zeroes at $\mathbb{Z}$.

$$\sum_{n=1}^{\infty}\left|\frac{z^2}{n^2}\right| = |z|^2\left(\frac{\pi^2}{6}\right) < \infty$$

Remark: $\prod (z-a_n)$

$$\prod \left(1-\frac{z}{a_n}\right) \leftarrow \text{better}$$

If $a_n \to \infty$ fast enough, can see conv.

EX: $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$ Ouch! $\sum_1^\infty \left|\frac{z}{n}\right| = |z|\sum_1^\infty \frac{1}{n} \to \infty$.

Weier: multiply by something here.

Inspiration: $1=(1-z)\,e^{-\operatorname{Log}(1-z)}$

$$-\operatorname{Log}(1-z) = z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots$$

$$P_N(z) = z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\frac{z^N}{N}$$

Can make $1 \approx (1-z)\,e^{P_N(z)}$ as close as desired on $D_{1/2}(0)$ by taking N big.

Weier: $f(z) = \prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right)e^{P_{N_n}\left(\frac{z}{a_n}\right)}\right]^{m_n}$

Famous formulas

$$\sin \pi z = \pi z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

$$= \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

both periodic with period 1.

Same principal parts at $\mathbb{Z}$

$H(z) = (\text{left}) - (\text{right})$ entire

both tend zero along vert lines $\to \pm\infty$.

H bndd entire. $\to 0$. So $\equiv 0$.