Lecture 41 The inhomogeneous Cauchy-Riemann eqns

Review problems on home page

The $\bar\partial$-operator

$$\begin{cases} dz = dx + i\,dy \\ d\bar z = dx - i\,dy \end{cases} \qquad dx = \tfrac{1}{2}(dz + d\bar z), \quad dy = \tfrac{1}{2i}(dz - d\bar z)$$

If $f = u + iv$, then $\frac{\partial f}{\partial x} = u_x + i v_x$ and $\frac{\partial f}{\partial y} = u_y + i v_y$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial z}} dz + \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial \bar z}} d\bar z$$

$\frac{\partial f}{\partial z} \leftarrow$ def^{n}s $\rightarrow \frac{\partial f}{\partial \bar z}$

Facts: $f = u + iv$. u & v satisfy the C-R eqns

$$\iff \frac{\partial f}{\partial \bar z} \equiv 0.$$

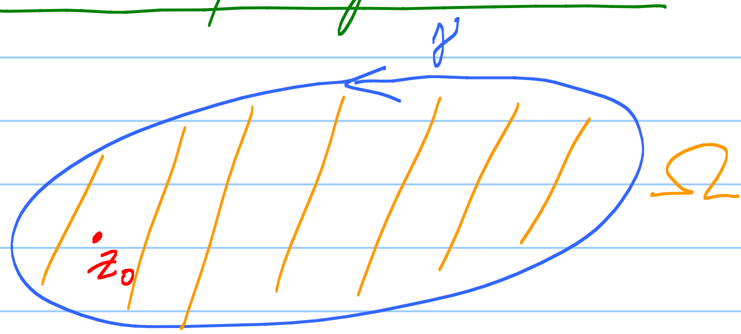
and, if f is analytic, then $\frac{\partial f}{\partial z} = f'$.

Complex chain rule

$w = f(\zeta)$ where $\zeta = g(z)$.

$$\begin{cases} \dfrac{\partial w}{\partial z} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial z} + \dfrac{\partial w}{\partial \bar\zeta}\dfrac{\partial \bar\zeta}{\partial z} \\[2ex] \dfrac{\partial w}{\partial \bar z} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial \bar z} + \dfrac{\partial w}{\partial \bar\zeta}\dfrac{\partial \bar\zeta}{\partial \bar z} \end{cases}$$

Improved Cauchy integral formula (Pompieu)



$$u(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{u(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial u}{\partial \bar z}(z)}{z-z_0}\,dz\wedge d\bar z$$

for any $u \in C^\infty(\bar\Omega)$. [$u$ complex valued]

Note: u analytic, then $\frac{\partial u}{\partial \bar z} \equiv 0$ and get CIF.

$$dz\wedge d\bar z = (dx + i\,dy)\wedge(dx - i\,dy)$$

$$= \underbrace{dx\wedge dx}_{0} - i\,dx\wedge dy + i\,\underbrace{dy\wedge dx}_{-dx\wedge dy} + \underbrace{dy\wedge dy}_{0}$$

$$= -2i\,dx\wedge dy \leftarrow dx\wedge dy \sim dx\,dy \text{ area measure.}$$

Theorem: (Lars Hörmander, Several complex variables, Chap 1)

Given a complex valued C^∞ fcn v on a domain Ω, there is a $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z} = v$.

Note: If u_1 and u_2 both sol^{n}s, then

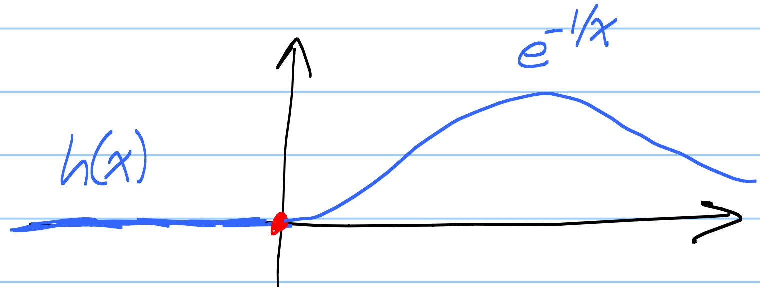
$\frac{\partial}{\partial \bar z}(u_1 - u_2) = v - v \equiv 0$ and $u_1 - u_2 = h$, an analytic fcn. Sol^{n} unique up to $+$(analytic fcn).

Cor: Mittag-Leffler thm holds on any domain Ω.

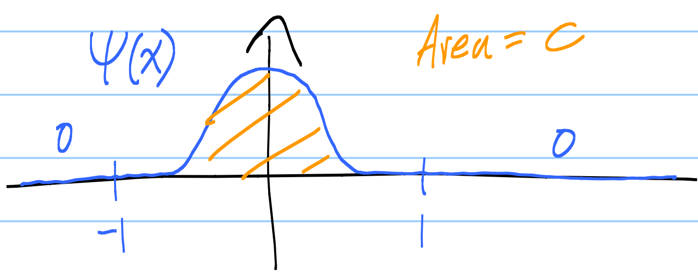
Given discrete $A = \{a_n\}_{n=1}^\infty \subset \Omega$ and princ parts $R_n(z)$ at each a_n, $\exists$ mero f fcn on Ω with poles only at pts in A with prescribed princ part.

<u>Pf</u>: Plan: Solve M-L prob in C^∞.
Correct it by solving a $\bar\partial$-problem.

<u>Step 0</u> Cook C^∞ "bump fcns" and cut-off fcns.



h is $C^\infty(\mathbb{R})$



$h(x+\frac{1}{2})\,h(-x+\frac{1}{2})$

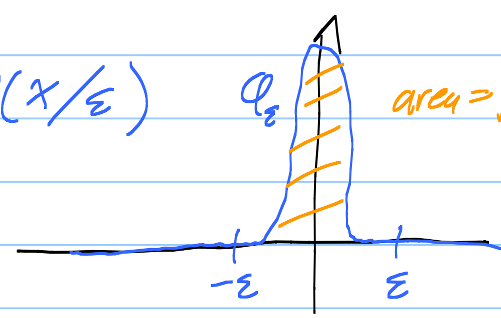
Support of ψ = Supp ψ = closure of $\{x : \psi(x) \neq 0\}$.
Supp ψ is <u>compact</u> in $(-1,1)$.

$$\varphi(x) = \frac{1}{c}\psi(x)$$

even, $\in C_0^\infty(-1,1)$

$$\int_{-\infty}^{\infty} \varphi \, dx = 1.$$

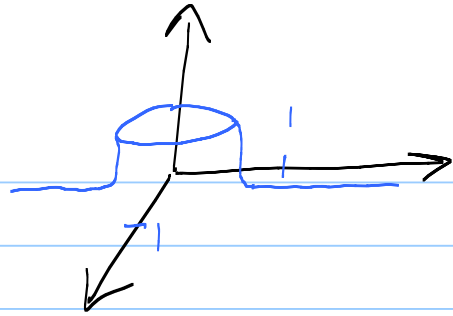
$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$$

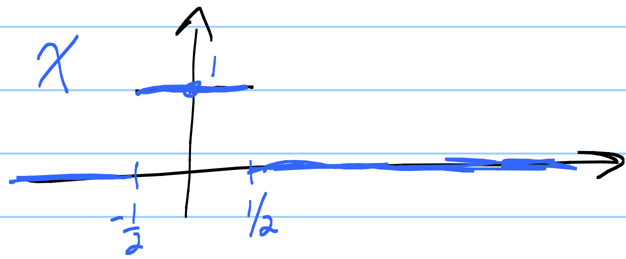


Classic fact: $\varphi_\varepsilon * f$ ← smooths out f

"close to f in L^2, etc"

C^∞ "birthday cake fcn"





$$\psi = (\varphi_\varepsilon * \chi)(x) = \int_{-\infty}^{\infty} \varphi_\varepsilon(x-y)\,\chi(y)\,dy$$

If $\varepsilon < \frac{1}{2}$, $\psi \in C_0^\infty(-1,1)$ and $\psi \equiv 1$ near 0.

Birthday cake fcn : $\chi_r^{z_0}(z) = \psi\left(\frac{|z-z_0|^2}{r^2}\right)$

centered at z_0, compact supported in $D_r(z_0)$, C^∞ smooth, $\equiv 1$ on a nbhd of z_0.

<u>Pf of M-L</u> : A discrete. So $\exists$ discs

$\overline{D_{r_n}(a_n)} \subset \Omega$ and $\overline{D_{r_n}(a_n)} \cap \overline{D_{r_m}(a_m)} = \emptyset$

when $n \neq m$. Let $U = \sum_{n=1}^{\infty} \chi_n R_n$ ← finite sum at any spot.

where χ_n is a C^∞ cut-off fcn suppored in $D_{r_n}(a_n)$

and $\equiv 1$ on a nbhd of a_n.

U is a "C^∞ solution to M-L prob"

Near a_n: $\quad U - R_n = (\chi_n R_n) - R_n$

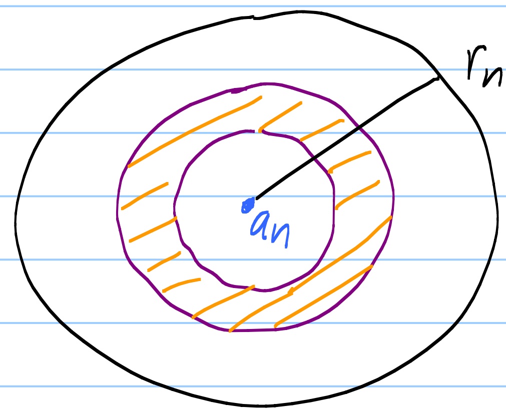
$\equiv 0$ near a_n.

Hmmm. $$\frac{\partial U}{\partial \bar{z}} = \sum_1^\infty \left(\underbrace{\frac{\partial \chi_n}{\partial \bar{z}}} R_n + \chi_n \frac{\partial R_n}{\partial \bar{z}} \right)$$

zero near a_n $\quad$ bad at a_n $\quad$ $\equiv$ zero (not defined at a_n)

Aha! Define $$v = \begin{cases} 0 & \text{at } a_n\text{'s} \\ \frac{\partial U}{\partial \bar{z}} & \text{on } \Omega - A \end{cases}$$

Support of $\frac{\partial U}{\partial \bar{z}}$



v is C^∞-smooth on Ω.

Get $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar{z}} = v$.

<u>Claim</u>: $f = U - u$ solves M-L prob!

<u>Why</u>: $\frac{\partial f}{\partial \bar{z}} = \frac{\partial U}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} = v - v \equiv 0 \quad z \neq a_n$.

f analytic on $\Omega - A$. ✓

Near a_n: $f - R_n = (\underbrace{U}_{\chi_n R_n} - u) - R_n$

$$= \underbrace{(1-\chi_n)R_n}_{\equiv 0 \text{ near } a_n} - \underset{\substack{\uparrow \\ \text{bndd}}}{u}$$

a_n is a removable singularity for $f - R_n$.