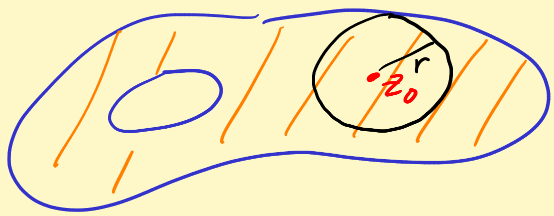
Lecture 9 Analytic functions are given by power series

Notation

$$\frac{1}{1-z} = \underbrace{1+z+\cdots+z^N}_{S_N(z)} + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

Showed that if $|z|<\rho<1$, then $|E_N(z)| < \frac{\rho^{N+1}}{1-\rho}$

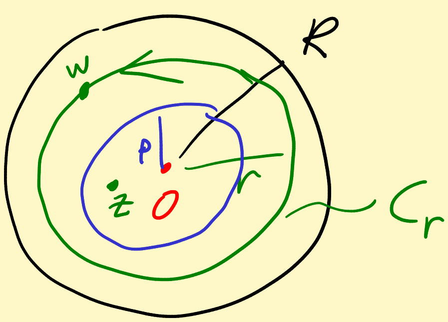
Thm Suppose f analytic on $D_R(z_0)$. Then
$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$ where
the R of C $\geq R$.



R of C $\geq r$

Pf: Can assume $z_0=0$ (otherwise $w=z-z_0$, etc.)

Let $0<\rho<r<R$



$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\underbrace{\frac{1}{1-\frac{z}{w}}}_{S_N\left(\frac{z}{w}\right)+E_N\left(\frac{z}{w}\right)}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\left(1+\left(\frac{z}{w}\right)+\cdots+\left(\frac{z}{w}\right)^N\right)dw + \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}E_N\left(\frac{z}{w}\right)dw}_{\mathcal{E}_N(z)}$$

$$= \sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{n+1}}\, dw \right)}_{\frac{f^{(n)}(0)}{n!}} z^n + \mathcal{E}_N(z)$$

and $$|\mathcal{E}_N(z)| \le \frac{1}{2\pi} \left(\max_{C_r} \frac{|f|}{r} \right) \frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}} \cdot \underbrace{\text{Length}(C_r)}_{2\pi r}$$

$$\le \frac{M \left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}} \quad \text{where } M = \max_{C_r} |f|$$

$$\longrightarrow 0 \text{ as } N \to \infty \text{ because } \frac{\rho}{r} < 1.$$

Get unif conv on $D_\rho(0)$ to f. Can squeeze $r, \rho \nearrow R$, we see power series converges on $D_R(0)$. So RofC $\ge R$.

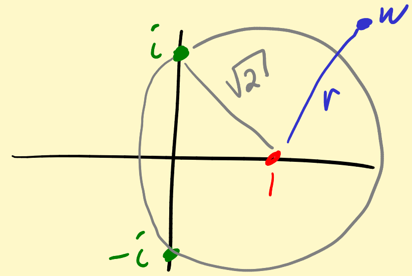
<u>Today</u> $e^z = "E(z)" = \sum_0^\infty \frac{z^n}{n!}$ R. of C. $= \infty$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $(R=\infty)$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$ $(R=\infty)$

$\tan z = \frac{\sin z}{\cos z}$ $R = ?$

<u>EX</u>: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$



$R \ge \sqrt{2}$

Claim $R \le \sqrt{2}$ too. $f(z) = \sum_0^\infty a_n (z-1)^n$ on $D_{\sqrt{2}}(1)$.

if P.S. converged at w. P.S. $\to F(z)$ analytic $D_r(1)$

Hmmm. $f(z) \equiv F(z)$ on $D_{\sqrt{2}}(1) \subset D_r(1)$.

blows up $\pm i$ doesn't! $\lightning$.

Thm: Can differentiate power series term by term, get same R of C as f for $f', f'', \dots$

Pf: f analytic on $D_R(z_0)$, $R = R$ of C.

Know f' analytic on $D_R(z_0)$ too.

So $R_{f'} \geq R_f$.

Claim: $R_{f'} \leq R_f$ too.

If p.s. for f' converged on bigger disc to $\tilde{f}$ analytic on bigger disc.

Let $F' = \tilde{f}$

$$\frac{d}{dz}\left[F - f\right] \equiv 0 \quad \text{on } D_{R_f}(z_0).$$

So $f = \underbrace{F + c}_{\substack{\text{bigger} \\ \text{R. of C.} \\ \text{than } f.}}$ on $D_{R_f}(z_0)$ $\lightning$

Or Use Hadamard's formula and $\lim_{n\to\infty} \sqrt[n]{n} = 1$

and Lim Sup facts.

End of pf about diff p.s. :

$$f'(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \quad \text{where}$$

$$b_n = \left. \frac{\frac{d^n}{dz^n}\left[f'(z)\right]}{n!} \right|_{z=z_0} = \frac{f^{(n+1)}(z_0)}{n!} \frac{(n+1)}{(n+1)}$$

$$= (n+1) \frac{f^{(n+1)}(z_0)}{(n+1)!}$$

$$= (n+1) a_{n+1} \checkmark$$

Next time: Zeroes of analytic fcns behave like zeroes of polynomials on open sets.

$$f(z) = \underset{\substack{\| \\ f(z_0) \\ \| \\ 0}}{a_0} + \underset{\substack{\| \\ \frac{f'(z_0)}{1!} \\ \| \\ 0}}{a_1}(z-z_0) + \cdots + \underset{\substack{\| \\ \frac{f^{(N)}(z_0)}{N!} \\ \| \\ 0}}{a_N}(z-z_0)^N + \underset{\substack{\uparrow \\ \text{not zero}}}{a_{N+1}}(z-z_0)^{N+1} + \cdots$$

$$= a_{N+1}(z-z_0)^{N+1} + a_{N+2}(z-z_0)^{N+2} + \cdots$$

$$= (z-z_0)^N \left[\underbrace{a_{N+1} + a_{N+2}(z-z_0) + a_{N+3}(z-z_0)^2 + \cdots}_{F(z)} \right]$$

Hmmm: Power series for $F(z)$ converges exactly for same z as p.s. for $f(z)$. So they have <u>same</u> R of C.

$$f(z) = (z-z_0)^{N+1} F(z)$$

Hmmm. $F(z)$ analytic on $D_{R_f}(z_0)$. $\Rightarrow$ F cont there

$$F(z_0) = a_{N+1} \neq 0$$

So F non-vanishing in a disc $D_\delta(z_0)$ $\quad \varepsilon = \frac{|a_{N+1}|}{2}$

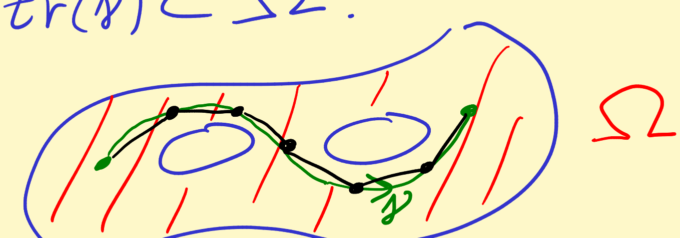
Conclude: z_0 is <u>only</u> zero of f in $D_\delta(z_0)$

z_0 is an <u>isolated</u> zero.

Connected open sets are called <u>domains</u>.

<u>Defn #1</u>: An open set Ω is <u>path-connected</u> if any two pts in Ω can be joined by a piecewise $\mathcal{C}^1$-curve γ, $tr(\gamma) \subset \Omega$.

<u>Remark</u>: Topology books



Can get piecewise $\mathcal{C}^1$ curve γ from a continuous one.

<u>Def^n #2</u>: Ω open, $\Omega = U_1 \cup U_2$ both open
(Ω connected)
$U_1 \cap U_2 = \emptyset$

Then $U_1 = \Omega$ and $U_2 = \emptyset$
<u>or</u> $U_1 = \emptyset$ and $U_2 = \Omega$

<u>Fact</u>: for open sets Def #1 equiv to #2.