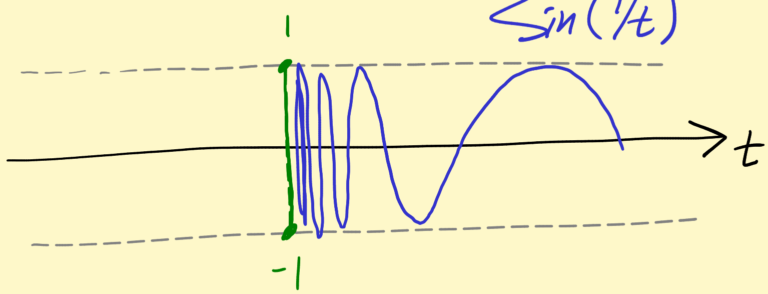
Lecture 10 Zeroes of analytic functions

(Connected) $\Longleftrightarrow$ (Path connected) for open sets in $\mathbb{C}$

$\Omega = U_1 \cup U_2$ open

$U_1 \cap U_2 = \emptyset$

... but not in general.



Connected but not path connected in relative top from $\mathbb{R}^2$

Missing piece from last lecture: $f(z) = \sum_0^\infty a_n z^n$, $R > 0$.

Then coeff a_n are uniquely determined: Taylor's formula.

Pf: P.S. converges uniformly on $\overline{D_r(0)}$, $r < R$. $0 < \rho < r < R$

$$f^{(N)}(0) = \frac{N!}{2\pi i} \int_{C_\rho(0)} \frac{f(z)}{z^{N+1}}\, dz = \sum_{n=0}^{\infty} \frac{N!}{2\pi i} \int_{C_\rho} \underbrace{\frac{a_n z^n}{z^{N+1}}}_{z^{n-N-1}} dz$$

all zero except $n = N$ case

(unif conv on $C_\rho \Rightarrow \int \sum = \sum \int$.)

$$= N!\, a_N \checkmark$$

Lemma If f is analytic on $D_r(z_0)$, $f(z_0) = 0$, then

either A) $f \equiv 0$ on $D_r(z_0)$ or

B) z_0 is an isolated zero of f.

Pf: $f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$

a_N ← first nonzero coeff in p.s.

$$= (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z),\quad F(z_0) = a_N \neq 0} \Big]$$

Def^n: N = order of zero at z_0. (multiplicity)

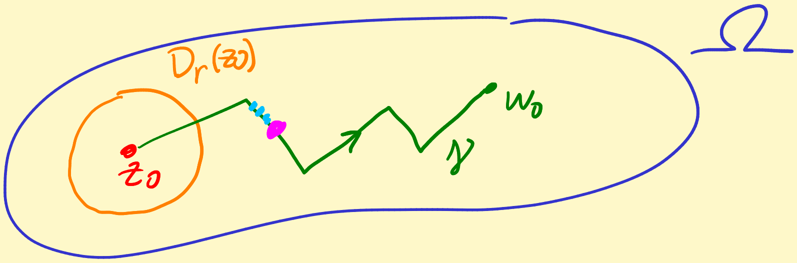
Thm (Identity Theorem) Suppose f analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$.

Then either A) $f \equiv 0$ on Ω or

B) Z_f has no limit points in Ω.

Pf: Suppose $z_0 \in \Omega$ is a limit pt of Z_f, meaning $\exists$ seq $\{z_n\}_{n=1}^{\infty}$ in Z_f, $z_n \neq z_0$ and $z_n \to z_0$ as $n \to \infty$. Note: f analytic $\Rightarrow$ f cont, so $z_n \to z_0 \Rightarrow \underbrace{f(z_n)}_{=0} \to f(z_0) = 0$.

Ω open $\Rightarrow \exists\, D_r(z_0) \subset \Omega$. z_0 is not an isolated zero. So Lemma $\Rightarrow f \equiv 0$ on $D_r(z_0)$. Pick $w_0 \in \Omega$. Ω path connected. So $\exists$ curve γ in Ω connecting z_0 to w_0. Can assume γ is piecewise linear.



$\gamma : z(t)\ ,\ a \le t \le b$

Let $T = \sup\{\tau : f(z(t)) = 0 \text{ for } a \le t < \tau\}$

Clear that $T > a$. Claim: $T = b$.

Easy to see $z(T)$ is <u>not</u> an <u>isolated</u> zero.

So $f \equiv 0$ on $D_\rho(z(T))$. Aha! T is not sup! $\lightning$

Alternate pf: $U_1 = \{\text{limit pts of } Z_f\}$. $z_0 \in U_1$.

$$U_2 = \Omega - U_1.$$

Use Lemma. See U_1 open. Show U_2 open.

$$U_1 \neq \emptyset \implies U_2 = \emptyset.$$

<u>Consequence</u>: e^z is unique analytic extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.

<u>Cor</u>: If $f(z)$ and $g(z)$ are analytic on a domain Ω and agree on set $\mathcal{O}$ with a limit pt in Ω, then $f \equiv g$ on Ω.

<u>Remark</u>: <u>Every</u> pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$ in $\mathbb{C}$.

Cosequence Trig identities hold for complex angles

$$\sin 2\theta = 2\sin\theta\cos\theta \quad \text{holds for } \theta \in \mathbb{C}.$$

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta \quad \begin{pmatrix} \text{first}, \ \alpha = z. \\ \text{then}, \ \beta = w. \end{pmatrix}$$

Identity thm way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} e^{-1/t^2}\sin(1/t) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

h is in $C^\infty(\mathbb{R})$ and vanishes to ∞ order at 0. Zeroes pile up at 0, but $h \not\equiv 0$.

Classic example of a C^∞ fcn $\neq$ its Taylor series, $z_0=0$.

Remark: Can factor out finitely many zeroes:

$$f(z) = (z-z_0)^N (z-w_0)^M \cdots (z-\zeta_0)^L H(z)$$

$$H(z) \neq 0, \quad z = z_0, w_0, \dots, \zeta_0.$$

EX: $\underbrace{\sin\frac{1}{z}}_{\substack{\text{zeroes at} \\ \{1/n\pi,\ n\in\mathbb{Z},\ n\neq 0\}}}$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin z = 0 \iff z = n\pi, \ n \in \mathbb{Z}.$$

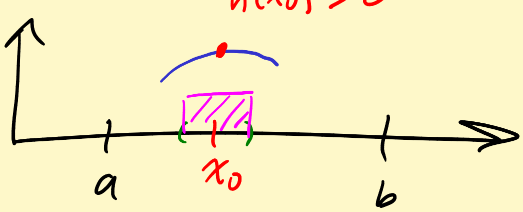
Aha! $\Omega = \mathbb{C} - \{0\}$ OK for zeroes to pile up at boundary pts of Ω.

Next time: Maximum principle.

Calculus lemma: $h:[a,b]\to\mathbb{R}$ continuous on $[a,b]$ and $h(t)\geq 0$ on $[a,b]$.

Then $\int_a^b h(t)\,dt = 0 \implies h\equiv 0$.

Pf:



$\varepsilon = \frac{h(x_0)}{2}$

$\int >$ area box. $\lightning$.

Lemma: $h:[a,b]\to\mathbb{R}$, $h(t)\leq M$ on $[a,b]$, continuous.

$$\int_a^b h(t)\,dt = M(b-a) \implies h(t)\equiv M \text{ on } [a,b].$$

Pf: Use $M-h$ in place of h in prev lemma