Lecture 12 Harmonic functions and harmonic conjugates

Def^n: A harmonic fcn on a domain is a real valued C^2-smooth fcn u such that $\Delta u \equiv 0$. $\left[\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]$

Important tool: If u is harmonic, $u_x - i u_y$ is analytic.

$U = u_x$, $V = -u_y$

Pf: U, V C^1-smooth.

C-R eqns: $\begin{cases} \frac{\partial U}{\partial x} \overset{?}{=} \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} \overset{?}{=} -\frac{\partial V}{\partial x} \end{cases}$ $\quad \frac{\partial}{\partial x}(u_x) \overset{?}{=} \frac{\partial}{\partial y}(-u_y)$ ✓ $\Delta u \equiv 0$

$\frac{\partial}{\partial y}(u_x) \overset{?}{=} -\frac{\partial}{\partial x}(-u_y)$ ✓

u C^2-smooth.
mixed partials =

Motivation: If u had a harmonic conjugate v

$f = u + iv$ analytic. $\quad f' = \begin{cases} u_x + i v_x \ \#1 \\ v_y - i u_y \ \#2 \end{cases} = \underline{u_x - i u_y}$

Cor: u C^2 smooth and harmonic $\Rightarrow$ u C^∞-smooth.

Cor: If u harmonic on a domain Ω and u vanishes on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Pf: $u_x - i u_y$ analytic on Ω, $\equiv 0$ on $D_\varepsilon(z_0)$.

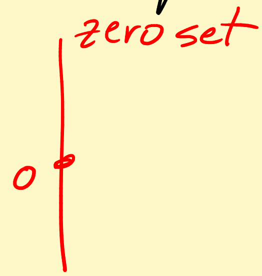
Identity thm $\Rightarrow$ $u_x - i u_y \equiv 0$ on Ω.

$\Rightarrow \nabla u \equiv 0$ on Ω. $\Rightarrow$ u const on Ω.

$u(z_0) = 0$, so const. $= 0$.

Can zeroes of harmonic fcns have limit pts? Yes!

Ex: $u(x,y) = \text{Re}(x+iy) = x$ harmoic.



Future Cauchy thm: If f is analytic on a simply connected (no holes) domain Ω and γ is a closed curve in Ω, then $\int_\gamma f\, dz = 0$.

Cor: If u is harmonic on a domain Ω that satisfies the Cauchy thm, u has a harm conj v on Ω.

Pf: Cauchy thm holds on convex open sets Ω.

Convex case: Hmmm: $f = u + iv$. $f' = \underbrace{u_x - iu_y}_{g}$

Get antiderivative F for g via

$$F(z) = \int_{L_a^z} g(w)\, dw$$

Write $F = U + iV$. Know $\underbrace{F'}_{U_x - iU_y} = g = u_x - iu_y$

Aha! $\nabla u \equiv \nabla U$. So $U = u + c$ on Ω, c const.

V is a harmonic conjugate for $u+c$, and hence for u too!

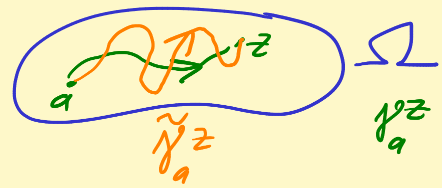
General case: Define

$$F(z) = \int_{\gamma_a^z} g(w)\, dw$$

where γ_a^z is a curve in Ω starting at a fixed pt a and going to z.

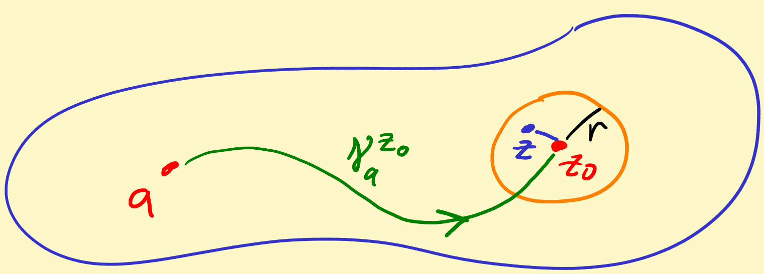
Step 1: F is well defined.

Pf: If $\tilde{\gamma}_a^z$, then $\underbrace{\gamma_a^z \cup (-\tilde{\gamma}_a^z)}_{\Gamma}$ is a closed curve in Ω.



$$0 = \int_\Gamma g\, dw = \int_{\gamma_a^z} g\, dw - \int_{\tilde{\gamma}_a^z} g\, dw \quad \checkmark$$

Step 2: Restrict attention to $D_r(z_0) \subset \Omega$.



Want to show $F'(z) = g(z)$

Aha! Take $F(z) = \int_{\gamma_a^{z_0} \cup L_{z_0}^z} g(w)\, dw$

$$= \underbrace{\int_{\gamma_a^{z_0}} g\, dw}_{\text{const.}} + \underbrace{\int_{L_{z_0}^z} g\, dw}_{\frac{d}{dz} = g}.$$

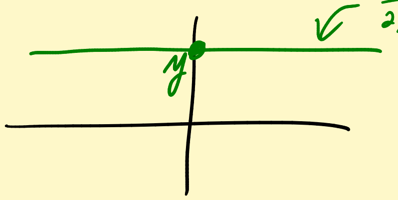
So $F' = g$ on $D_r(z_0) \leftarrow$ convex.

Cooking up harmonic conjugates on $D_R(0)$ or $\mathbb{C}$.

Lemma: $\dfrac{\partial u}{\partial x} \equiv 0 \iff u =$ fcn of y alone

$\dfrac{\partial u}{\partial y} \equiv 0 \iff u =$ fcn of x alone.

Pf:



✓ $\frac{\partial u}{\partial x} \equiv 0 \Rightarrow u$ const on horiz line

$u = C(y)$

Cor: $\dfrac{\partial u_1}{\partial x} \equiv \dfrac{\partial u_2}{\partial x} \Rightarrow u_1 = u_2 + (\text{arbitrary fcn of } y)$

$\frac{\partial}{\partial y}$ too

Prob: Find a harmonic conj for $u = \underbrace{x^2 - y^2 + x + y}_{\text{harmonic ✓}}$ on $\mathbb{C}$.

C-R eqns: Find v with

$u_x = v_y$ $\qquad$ $v_x = -u_y = 2y + 0 - 1 = 2y - 1$ (A)

$u_y = -v_x$ $\qquad$ $v_y = u_x = 2x - 0 + 1 + 0 = 2x + 1$ (B)

First use (A): $V = \int 2y - 1 \, dx = \underbrace{2xy - x}_{v} + \underbrace{C(y)}_{\text{arb fcn of } y.}$

"partial integral" treat y like a constant

[Know $\frac{\partial v}{\partial x} = \frac{\partial V}{\partial x} = 2xy - 1$. So they diff by fcn of y. ✓]

Next use (B) in different way to pin down $C(y)$

$$(B) \quad \frac{\partial v}{\partial y} = 2x+1$$

$$\frac{\partial}{\partial y}\left(\underbrace{2xy - x + C(y)}_{\text{from (A)}}\right) \overset{\text{want}}{=} 2x+1$$

$$2x - 0 + C'(y) = 2x + 1$$

Miracle: x's cancel out.

$$C'(y) = 1$$

So $C(y) = y + K$.

So $v = 2xy - x + (y+K)$ ← uniq up to $+K$.

Danger of having holes in Ω. $\operatorname{Ln}|z| + i\operatorname{Arg} z$ is analytic on $\mathbb{C} - (-\infty, 0]$. $\operatorname{Ln}|z|$ is harmonic on $\underbrace{\mathbb{C} - \{0\}}_{\Omega}$. Suppose v is a harmonic conjugate for $\operatorname{Ln}|z|$ on Ω. Then v and $\operatorname{Arg} z$ are both harmonic conjugates for $\operatorname{Ln}|z|$ on $\mathbb{C} - (-\infty, 0]$.

So $\underbrace{v}_{\text{no jump!}} = \underbrace{\operatorname{Arg} z + K}_{\text{jump at slit}}$ there. $\lightning$.