Lecture 15 Singularities at ∞, Partial fractions

If f is analytic on $D_\varepsilon(\infty) - \{\infty\} = \{z : |z| > \frac{1}{\varepsilon}\}$ for some $\varepsilon > 0$, we say f has "an isolated singularity at ∞."

Def^n Type of singularity of $f(z)$ at ∞ is the type of the singularity of $f(1/z)$ at $z=0$.

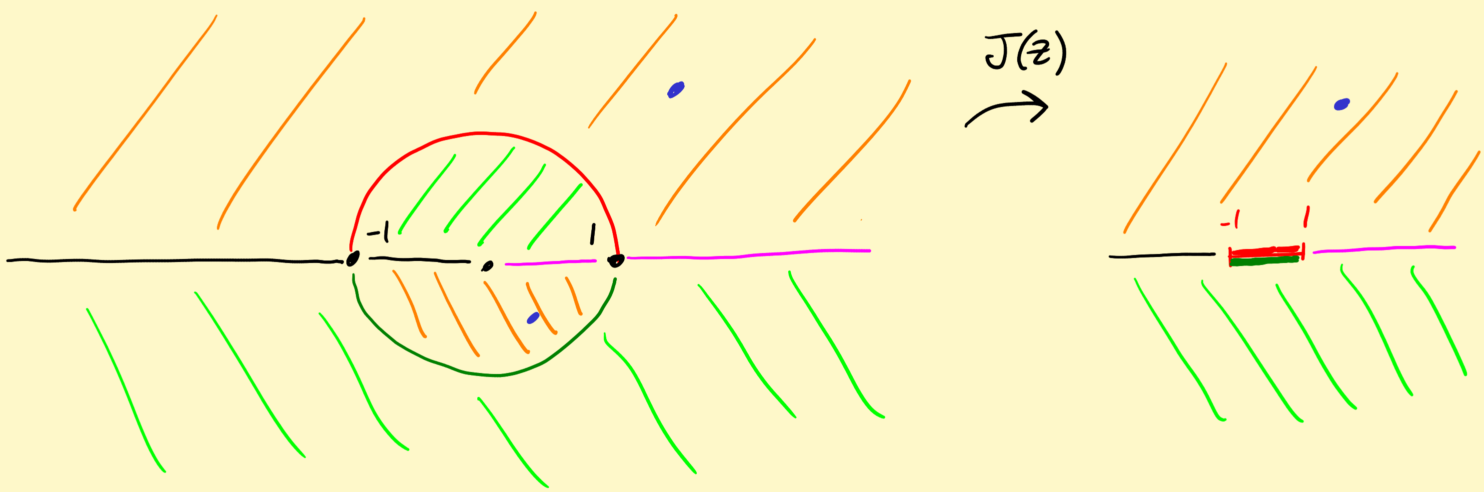
EX: Entire fcns have singularities at ∞.

e^z has an ess. sing. at ∞ because $e^{1/z}$ has one at $z=0$.

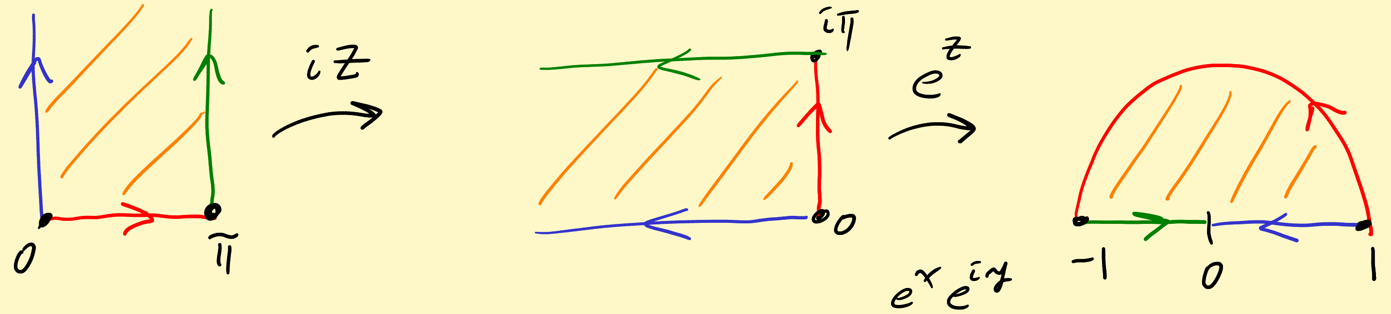
Fact: Entire fcn has

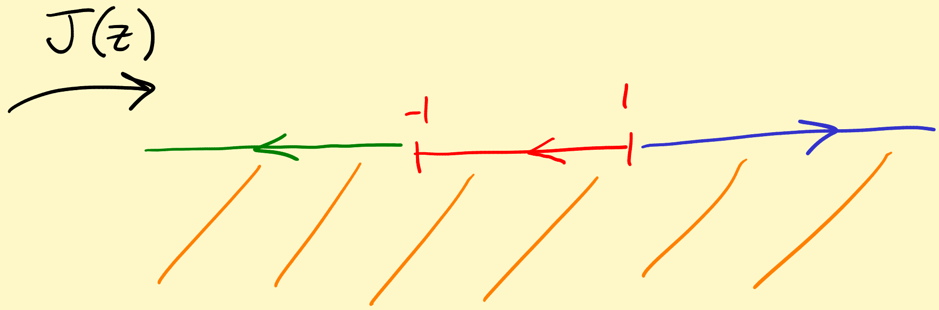
1) a removable sing at ∞ $\iff$ it's $\equiv$ const.
2) a pole at ∞ $\iff$ it's a poly
   (order of pole at ∞) = degree
3) Everything else has an ess sing at ∞.

Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

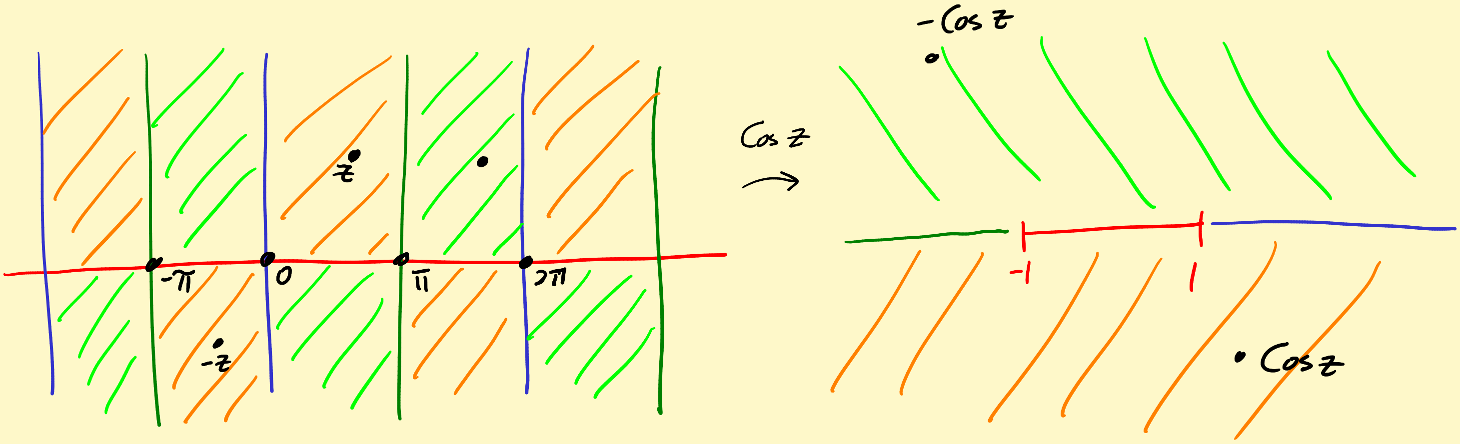


Complex cosine $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right) = J(e^{iz})$





composition is $\cos z$



$$\cos(-z) = \cos z$$

$$\cos(z+\pi) = -\cos z$$

See $\cos z : \mathbb{C} \to \mathbb{C}$ ∞-to-one

Partial fractions decomposition. Suppose $P(z)$, $Q(z)$ polys and $N_P = \deg P < N_Q = \deg Q$. Then

$\dfrac{P(z)}{Q(z)}$ has a part. frac. decomp as follows:

$$Q(z) = A\,(z-r_1)^{m_1}(z-r_2)^{m_2}\cdots(z-r_n)^{m_n}$$

where $\sum_{k=1}^{n} m_k = N_Q$.

Near r_k : $\dfrac{P(z)}{Q(z)} = \dfrac{1}{(z-r_k)^{m_k}} \left[\dfrac{P(z)}{\underbrace{q_k(z)}_{\text{poly},\ \neq 0 \text{ at } r_k}} \right]$

poly, $\neq 0$ at r_k.

$\underbrace{\qquad\qquad}_{A_0 + A_1(z-r_k) + \cdots}$

$$= \underbrace{\frac{A_0}{(z-r_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-r_k}}_{\text{principal part at } r_k} + \left(\begin{array}{l} \text{power series} \\ \text{analytic near } r_k \end{array} \right)$$

principal part at r_k

$= R_k$

<u>Claim</u>: $\dfrac{P}{Q} = \sum_{k=1}^{n} R_k$ ← part. frac. decomp.

<u>Pf</u>: Let $f = \dfrac{P}{Q} - \sum_{k=1}^{n} R_k$

<u>Claim</u>: f has removable sing at each r_k and when given proper values at r_k, is a <u>bdd</u> entire fcn!

Near r_k : $f = \underbrace{\left(\dfrac{P}{Q} - R_k\right)}_{\substack{\text{removable} \\ \text{sing at } r_k}} - \underbrace{\sum_{j \neq k} R_j}_{\substack{\text{analytic} \\ \text{near } r_k}}$ ✓

<u>Claim</u>: $f(z) \to 0$ as $|z| \to \infty$.

Clear that each $R_k(z) \to 0$ as $|z| \to \infty$.

Basic poly estimate: $A|z|^{N_P} \le \underline{|P(z)| \le B|z|^{N_P}}$, $|z| > R_P$

$\underline{a|z|^{N_Q} \le |Q(z)|} \le b|z|^{N_Q}$, $|z| > R_Q$

$$\left| \frac{P(z)}{Q(z)} \right| \le \frac{B|z|^{N_P}}{a|z|^{N_Q}} = \frac{B}{a} \frac{1}{|z|^{\underbrace{N_Q - N_P}_{>0}}} \quad \text{if } |z| > \max(R_P, R_Q)$$

$\to 0$ as $|z| \to \infty$.

As in pf of Fund thm alg, f is a bdd entire fcn.

Liouville's $\Rightarrow$ $f \equiv$ const.

$\to 0$ as $|z| \to \infty$ $\Rightarrow$ $f \equiv 0$. ✓

Freshman calculus: $\dfrac{x^5 + x^3 + x + 1}{(x-1)(x-2)^3(x^2+2x+2)}$

$$= \frac{A}{x-1} + \underbrace{\frac{B}{(x-2)^2} + \frac{C}{x-2}}_{} + \underbrace{\frac{Dx+E}{x^2+2x+2}}_{\dfrac{F}{x-r} + \dfrac{\overline{F}}{x-\overline{r}}}$$

<u>Schwarz lemma</u> Suppose f is analytic on $D_1(0)$ and

$f : D_1(0) \to D_1(0)$ with $\underline{f(0) = 0}$.

Then $\quad$ A) $\quad |f(z)| \le |z| \quad$ for $z \in D_1(0)$

B) $\quad |f'(0)| \le 1$

<u>and</u> if equality occurs in (A) for <u>some $z \neq 0$</u>,

<u>or</u> if equality occurs in (B), then

$$f(z) = \underbrace{\lambda z}_{\text{rotation by } \infty.} \quad \text{for some unimodular const } \lambda = e^{i\theta}.$$